Field Theoretic Functional Calculus for the Anharmonic Oscillator in Low Approximations*

WOLFGANG SCHULER

Institut für Theoretische Physik University of Tübingen

(Z. Naturforsch. **23 a**, 1869—1887 [1968]; received 8 August 1968)

The theory of solution for quantum field functional equations as developped in II and III for a suitable test problem of quantum mechanics is investigated in low approximations. In Sect. 1 the functional formulation of the anharmonic oscillator is once more given and in Sect. 2 general translational equivalent functional equations. The expansion of the physical state functional into series of unsymmetrical and symmetrical base functionals and the representation of the functional equations for such expansions are discussed in Sect. 3. In the next Sect. 4 the unsymmetrical DYSON representation is investigated and the explicit representation of the smeared out functional equation by an infinite system of equations is derived. Then in Sect. 5 and 6 the system of equations is truncated for $N = 3$ and the corresponding eigenvalue equation is considered. The same is done in Sect. 7 and 8 for the HERMITTE representation. In the following Sect. 9 the original functional equation in a not smeared out form is treated in the DYSON representation and the corresponding system of unsymmetrized equations is given. Furthermore in Sect. 10 the $N = 3$ approximation together with other possibilities is investigated again. Finally the numerical results of our calculations for eigenvalues are stated and discussed. In the appendices technical details are derived.

Introduction

In nonlinear spinor theory the dynamical behaviour of elementary particles can be described by functionals of field operators in a HEISENBERG-representation and corresponding functional equations[1-3]. In configuration space the functional equations lead to infinite sets of differential or integral equations between the different matrix elements of field operators[4]. This description is of special interest, because it is formally valid for canonical as well as noncanonical quantization, where the usual SCHRÖDINGER theory is inapplicable[2]. However, up to now no systematic method of solutions has been given for these fieldtheoretic functional equations in the case of strong coupling.

For this the so-called NEW-TAMM-DANCOFF-method (N.T.D.-method) proposed by HEISENBERG[5] and first introduced by DYSON[6] can be generalized to an approximation scheme for the solution of the functional equations. In order to test this proposal the anharmonic oscillator is offered as the simplest example, the functional equations of which are analogous to those of nonlinear spinor theory, as is shown in [2] and [3]. This model has already been investigated in some papers. At first HEISENBERG calculated the lowest approximations of the one-time N.T.D.-method in the p-q-representation[7]. The numerical results were rather good in contrast to those of SCHWARTZ's[8], who did not properly take into account the commutation relations and f-sum rules as KAISER[9] had shown. Later STUMPF, WAGNER and WAHL[10] and WAGNER[11] proved the convergence of the one-time N.T.D.-method in the q-representation. They also got fairly good results.

However, the full functional analogy of the test system to nonlinear spinor theory requires the investigation of many-time functional equations. First results about the solution procedure for the one-time limiting case of the many-time functional-equations have been given in [12]. The systematic treatment of many-time functionals has been undertaken by

* This work is part of the thesis of the author at the University of Tübingen, 1968.

1 W. HEISENBERG, An Introduction to the Unified Theory of Elementary Particles, Wiley and Sons, London 1967.

2 H. RAMPACHER, H. STUMPF, and F. WAGNER, Fortschr. Phys. **13**, 385 [1965].

3 H. P. DÜRR and F. WAGNER, Nuovo Cim. **46**, 223 [1966].

4 E. FREESE, Z. Naturforschg. **8a**, 776 [1953].

5 W. HEISENBERG, Z. Naturforschg. **9a**, 292 [1954].

6 F. J. DYSON, Phys. Rev. **90**, 994 [1953]; **91**, 421 [1953]; **91**, 1543 [1953].

7 W. HEISENBERG, Nachr. Gött. Akad. Wiss. 111 [1953].

8 CH. SCHWARTZ, Ann. Physics **32**, 277 [1965].

9 H. J. KAISER, Ann. Phys. **6**, 131 [1960].

10 H. STUMPF, F. WAGNER, and F. WAHL, Z. Naturforschg. **19a**, 1254 [1964].

11 F. WAGNER, Thesis, University of Munich 1966.

12 D. MAISON and H. STUMPF, Z. Naturforschg. **21a**, 1829 [1966], named I.

Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung") beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition "no derivative works"). This is to allow reuse in the area of future scientific usage.

SCHULER and STUMPF in [13] and [14]. The general idea for the solution of the functional equations in [13] and [14] is the use of an expansion of the physical functionals into series of suitably chosen base functionals and to approximate the exact infinite series by series with a finite number of terms. As has been shown in [12], the approximation procedure can be performed either in a symmetrical or in an unsymmetrical functional operator representation. The unsymmetrical representation leads with the N.T.D. solution procedure to the wellknown field theoretic formulation and has been discussed in [13] while the symmetrical one has been investigated in [14]. This last representation seems to be more advantageous because symmetry properties of the functional operator such as formal hermiticty and selfadjointness are visibly preserved. Moreover, only in this case the convergence of the eigenvalues in the one-time limit could be proved [12] and also the proof of convergence given in [11] takes actually place in this representation. Naturally, in both representations we have the choice between the q or p-q-form of the matrixelements. Thus, because of simplicity, we have investigated in [13] the q version in the unsymmetrical functional representation, while in [14] the q-p version of the symmetrical representation has been discussed in order to have more analogy to nonlinear spinor theory. — Only the lowest numerical approximations have been given there.

It is the aim of this paper to compare numerically both possible functional representations in the q-version for the lowest states of odd parity of the anharmonic oscillator. Both the systematic approximation and other possibilities for approximation of the occuring equations in the lowest N.T.D.-step will be investigated.

I. Functional Representation

1. The Functional Equation

The equations of motion for the anharmonic oscillator are

$$\frac{\mathrm{d}}{\mathrm{d}t}\, q(t) = p(t)\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\, p(t) = -\, q^3(t) \qquad (1.1)$$

with the canonical commutation relation

$$[p(t), q(t)]_- = -\, i \cdot \mathbf{1}\,. \qquad (1.2)$$

The dynamical behaviour of the system (1.1) and (1.2) can be described by the set of time-ordered matrix elements[2].

$$\tau_n^{\varrho}(t_1 \dots t_n) = :\langle 0 \,|\, T\, q(t_1) \dots q(t_n) \,|\, \Psi_{\varrho}\rangle \qquad (1.3)$$

where T means time ordering and $|\,\Psi_{\varrho}\rangle$ is a stationary state of the anharmonic oscillator i.e. an eigenstate of the time translation operator H. $|0\rangle$ is the physical groundstate normed to unity. For their calculation an auxiliary space, the so-called functional space, is introduced, where the set of τ-functions is represented by a functional in the following form

$$\mathscr{T}_{\varrho}(j) = :\sum_{k=1}^{\infty} \int \tau_k^{\varrho}(t_1 \dots t_k)\, F_k(t_1 \dots t_k; j)\, \mathrm{d}t_1 \dots dt_k\,. \qquad (1.4)$$

The base functionals in the expansion (1.4) are defined by

$$F_k(t_1 \dots t_k; j) = :\frac{i^k}{k!}\, j(t_1) \dots j(t_k) \qquad (1.5)$$

with classical source functions $j(t)$. Observing (1.4) and (1.5) and the definition (1.3) the functional (1.4) may also be written

$$\mathscr{T}_{\varrho}(j) = \langle 0 \,|\, T \exp[i \textstyle\int q(t)\, j(t)\, \mathrm{d}t] \,|\, \Psi_{\varrho}\rangle\,. \qquad (1.6)$$

For functionals one is able to define under certain conditions a functional differentiation and a functional integration[15–18]. Especially by functional differentiation one gets the τ_n-functions (1.3) back from (1.6)

$$\tau_n^{\varrho}(t_1 \dots t_n) = \frac{1}{i^n} \frac{\delta^n}{\delta j(t_1) \dots \delta j(t_n)}\, \mathscr{T}_{\varrho}(j)\big|_{j=0} \qquad (1.7)$$

Then, one can derive with (1.7) from the dynamical eqs. (1.1) and with (1.4) a functional equation characterizing $\mathscr{T}_{\varrho}(j)$. For details of its derivation we refer to II. One obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\delta}{\delta j(t)}\, \mathscr{T}_{\varrho}(j) = U\left(j(t), \frac{\delta}{\delta j(t)}\right) \mathscr{T}_{\varrho}(j) \qquad (1.8)$$

[13] W. SCHULER and H. STUMPF, Z. Naturforschg. **22a**, 1842 [1967], named II.

[14] W. SCHULER and H. STUMPF, Z. Naturforschg. **23a**, 902 [1968], named III.

[15] D. MAISON, Thesis, University of Munich 1967.

[16] K. SYMMANZIK, Z. Naturforschg. **9a**, 809 [1954], Appendix.

[17] Y. V. NOVOZHILOV and A. V. TULUB, The Method of Functionals in the Quantum Theory of Fields, Gordon and Breach, New York 1961.

[18] a) I. FRIEDRICHS and A. SHAPIRO, Seminar on Integration of Functionals, Inst. of Math. Science, University, New York 1957. — b) E. A. BEREZIN, The Method of Second Quantization, Academic Press, New York 1966. — c) L. GROSS, in Analysis in Function Spaces, Ed. W. T. MARTIN and I. SEGAL, MIT-Press, Cambridge, Mass. 1963.
— d) L. STREIT, in Quantumelectrodynamics, Ed. P. URBAN, Acta Physica Austriaca, Suppl. II, Wien 1965.

with $$U\left(j(t), \frac{\delta}{\delta j(t)}\right) = :\frac{\delta^3}{\delta j^3(t)} + i\,j(t)\,. \tag{1.9}$$

Additionally stationary functionals have to satisfy the subsidiary condition (1.10)

$$\mathbf{P}\cdot\mathscr{T}_\varrho(j) = :\int j(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\delta}{\delta j(t)}\,\mathrm{d}t\,\mathscr{T}_\varrho(j) = -i\,\omega_\varrho\,\mathscr{T}_\varrho(j)$$

with $\omega_\varrho = :(E_\varrho - E_0)$ and E_ϱ being the energy eigenvalue to the state $|\Psi_\varrho\rangle$.

2. Translational Equivalent Equations

It is convenient to introduce normal ordering of the interaction term by adding on both sides of (1.8) a contraction term[2], resulting in (2.1)

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3\,F(0)\,\varkappa\right]\frac{\delta}{\delta j(t)}\,\mathscr{T}_\varrho(j) = N\left(j(t)\,,\ \frac{\delta}{\delta j(t)}\right)\mathscr{T}_\varrho(j)$$

with (2.2)

$$N\left(j(t),\ \frac{\delta}{\delta j(t)}\right) = :U\left(j(t),\ \frac{\delta}{\delta j(t)}\right) + 3\,F(0)\,\varkappa\,\frac{\delta}{\delta j(t)}$$

where $F(0)$ is the vacuum expectation value of $q^2(0)$. Defining the Feynman-Green function for (2.1) by

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 3\,F(0)\,\varkappa\right]G(t-t') = \delta(t-t') \tag{2.3}$$

we get by application of G on (2.1) the functional equation

$$\mathbf{0}\left(j(t),\ \frac{\delta}{\delta j(t)}\right)\mathscr{T}_\varrho(j) \tag{2.4}$$
$$\equiv\left[\frac{\delta}{\delta j(t)} - \int G(t-t')\,N\left(j(t'),\ \frac{\delta}{\delta j(t')}\right)\mathrm{d}t'\right]\mathscr{T}_\varrho(j) = 0.$$

The parameter $\varkappa$ is different for various representations and will be fixed in chapter III.

However, equation (2.4) is not yet satisfactory, since the physical solutions of (2.4) do not depend explicitly on the parameter t i.e. for any arbitrary value of the parameter t we obtain the same solutions [14]. Therefore, we are allowed to use instead of (2.4) a suitably chosen linear combination over t for the calculation of $\mathscr{T}_\varrho(j)$ i.e. a smeared out equation. As discussed in III the variety of possible combinations is restricted by general requirements.

1. The linear combination has to maintain the symmetry of the resulting equations in all arguments.
2. The linear combined equation has to commute with **P**.
3. The linear combination is to be integrable.
4. The linear combination has to maintain formal Hermiticity of the operator **0**.

The condition 4. is of great importance because the property of formal Hermiticity of the functional equation (1.8) will probably enable us to prove convergence. In the q-representation it is impossible to fulfil condition 4 without neglecting condition 1 or 2. Therefore we ignore it here. The most general smearing out operator satisfying condition 2 reads

$$\mathbf{S} = :\int\left[j(t')\,s_1(t'-t) + \frac{\delta}{\delta j(t')}\,s_2(t'-t)\right]\mathrm{d}t\,\mathrm{d}t'. \tag{2.5}$$

Applying **S** to (2.4) we get the equation

$$\int\left[j(t')\,s_1(t'-t) + \frac{\delta}{\delta j(t)}\,s_2(t'-t)\right] \tag{2.6}$$
$$\times\,\mathbf{0}\left(j(t),\ \frac{\delta}{\delta j(t')}\right)\mathrm{d}t\,\mathrm{d}t'\,\mathscr{T}_\varrho(j) = 0\,.$$

To satisfy the conditions 1 and 3 the not yet determined functions $s_1(t)$ and $s_2(t)$ have to be specified which will be done in the next chapters. Thus we shall use the linear combined equation (2.6) instead of equation (2.4).

3. Functional Representations

For the practical construction of state functionals we have to expand them into series of suitably chosen base functionals and to truncate these series. For this purpose we use the socalled Dyson-base functionals

$$D_n(t_1\ldots t_n) \tag{3.1}$$
$$=: F_n(t_1\ldots t_n)\,\exp\left[-\tfrac{1}{2}\int j(\xi)\,F(\xi-\eta)\,j(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right]$$

with the two-point function

$$F(t_1-t_2) = :\langle 0\,|\,T\,q(t_1)\,q(t_2)\,|\,0\rangle\,. \tag{3.2}$$

These functionals have first been introduced by Dyson[6] and been extensively used in I, II and III. We then assume $\mathscr{T}_\varrho(j)$ to have the expansion (3.3)

$$\mathscr{T}_\varrho(j) = \sum_{n=1}^{\infty}\int\varphi_n(t_1\ldots t_n)\,D_n(t_1\ldots t_n;\,j)\,\mathrm{d}t_1\ldots\mathrm{d}t_n\,.$$

In order to calculate the expansion coefficients φ_n we must use the corresponding matrix representations of the functional equations. Then we can distinguish between two possible matrix representations. After substitution of the state functional expansion (3.3) in the functional equations we project

1. in the *unsymmetrical representation* the equation on the functionals $D^k(t_1\ldots t_k;\,j)$ being dual to base functionals $D_n(t_1\ldots t_n;\,j)$. These are defined by the condition

$$\int D^k(t_1'\ldots t_k';\,j)\,D_n(t_1\ldots t_n;\,j)\,\delta j \tag{3.4}$$
$$= \delta_{n,k}\,\frac{1}{n!}\,P\sum_{\lambda_1\ldots\lambda_n=1}^{n}\prod_{i=1}^{n}\delta(t_{\lambda_i}-t_i')\,.$$

The other possibility is

2. the *symmetrical representation*, where the functional equations are projected on to the original base functionals instead of the dual set. As the Dyson functionals are not orthogonal to each other, it is more advantageous to use in the symmetrical representation the so-called Hermite functionals $J_n(t_1 \dots t_n; j)$ defined in III. They are orthogonal to each other i.e.

$$\int J_m(t_1' \dots t_m'; j)\, J_n(t_1 \dots t_n; j)\, \delta j = \delta_{m,n} \frac{1}{n!} P \sum_{\lambda_1 \dots \lambda_n = 1}^{n} \prod_{i=1}^{n} \delta(t_{\lambda_i} - t_i'). \tag{3.5}$$

The expansion of $\mathscr{T}_\varrho(j)$ in Hermite functionals then reads

$$\mathscr{T}_\varrho(j) = \sum_{n=1}^{\infty} \int \eta_n(t_1 \dots t_n) \times J_n(t_1 \dots t_n; j)\, \mathrm{d}t_1 \dots \mathrm{d}t_n \frac{1}{\sqrt{n!}}. \tag{3.6}$$

In the symmetrical representation the expansions (3.3) and (3.6) are equivalent and connected with each other by a similarity transformation, stated in III.

It is very impotant, for the condition of stationarity (1.10) to have the same structure in both representations, namely in configuration space

$$\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t_i} \varphi_n(t_1 \dots t_n) = -i\,\omega_\varrho\, \varphi_n(t_1 \dots t_n) \tag{3.7}$$

and in Fourier space (a tilde denoting Fourier transforms)

$$\left[\sum_{i=1}^{n} q_i\right] \tilde{\varphi}_n(q_1 \dots q_n) = \omega_\varrho\, \tilde{\varphi}_n(q_1 \dots q_n). \tag{3.8}$$

These formulae are derived in II and III for the unsymmetrical and the symmetrical representation. Using for $\tilde{\varphi}_n(q_1 \dots q_n)$ the ansatz

$$\tilde{\varphi}_n(q_1 \dots q_n) = \delta\left(\sum_{i=1}^{n} q_i - \omega_\varrho\right) \tilde{\Psi}_n(q_1 \dots q_n). \tag{3.9}$$

with an arbitrary function $\tilde{\Psi}_n(q_1 \dots q_n)$ still to be determined the subsidiary condition (3.8) is automatically fulfilled and thus the centre of gravity is separated. For $n = 1$ the ansatz (3.9) is always satisfied, since we have

$$\varphi_1(t) \equiv \tau_1(t) = \langle 0 | q(t) | \Psi_\varrho \rangle = c_0\, e^{i\omega_\varrho t} \tag{3.10}$$

and in Fourier space

$$\tilde{\varphi}_1(p) = c_0 \cdot \delta(p - \omega_\varrho). \tag{3.11}$$

The same applies to the η-functions.

In the following chapters we shall investigate the unsymmetrical Dyson-representation and the symmetrical Hermite-representation, which have already been discussed in general in II for the q-version and in III for the $p-q$-version.

II. The Dyson Representation

4. *The φ-equations*

In order to get the unsymmetrical Dyson matrix representation we formally introduce the set $D^k(j)$ dual to the base functionals $D_n(j)$ by

$$D^k(t_1 \dots t_k; j) =: F^k(t_1 \dots t_k; j) \times \exp\left[\tfrac{1}{2} \int j(\xi) F(\xi - \eta) j(\eta)\, \mathrm{d}\eta\right]. \tag{4.1}$$

They fulfil formally the relation (3.4) and we then have

$$\int D^k(t_1 \dots t_k; j)\, \mathscr{T}_\varrho(j)\, \delta j = \varphi_k(t_1 \dots t_k). \tag{4.2}$$

The $F^k(t_1 \dots t_k; j)$ are dual to the $F_n(t_1 \dots t_n; j)$ defined by (1.5) and satisfy a relation of orthogonality analogous to (3.4). The functional integration — marked by the symbol δj — is meant here only formally and is defined by

$$\int F^k(t_1 \dots t_k; j)\, F_n(t_1' \dots t_n'; j)\, \delta j =: \frac{1}{i^n} \frac{\delta^k}{\delta j(t_1) \dots \delta j(t_k)} F_n(t_1 \dots t_n; j)\,|_{\,j\,=\,0}. \tag{4.3}$$

It is easily to be seen that the definition (4.3) only results in comparison of the same $j(t)$ power's coefficients. In the Dyson representation we have to choose the smearing out operator S for the symmetrized functional equation (2.6) to be

$$\mathbf{S} =: \int j(t')\, \delta(t' - t)\, \mathrm{d}t\, \mathrm{d}t'. \tag{4.4}$$

This choice fulfils the conditions 1 and 3 of Chapter I.2, namely the symmetry of the variables and the integrability, as is shown in II. Then Eq. (2.6) reads, already transformed into Fourier space (Fourier transforms are marked by a tilde).

$$\begin{aligned} m\, \tilde{\varphi}_m(q_1 \dots q_m) = & \\ & - \sum_{\lambda_1 = 1}^{m} \tilde{G}(q_{\lambda_1}) \int \overset{1}{\varphi}_{m+2}(q_{\lambda_1} - \eta \,|\, \eta\, q_{\lambda_2} \dots q_{\lambda_m}) \frac{\mathrm{d}\eta}{2\pi} \\ & + \sum_{(\lambda_1, \lambda_2) = 1}^{m} K(q_{\lambda_1} q_{\lambda_2})\, \overset{1}{\varphi}_m(q_{\lambda_1} + q_{\lambda_2} \,|\, q_{\lambda_3} \dots q_{\lambda_m}) \\ & + r_m(q_1 \dots q_m) \quad (m = 1 \dots \infty) \end{aligned} \tag{4.5}$$

with

$$r_m(q_1 \dots q_m) \tag{4.6}$$
$$=: \sum_{(\lambda_1,\lambda_2)=1}^{m} h(q_{\lambda_1})\, 2\pi\,\delta(q_{\lambda_1}+q_{\lambda_2})\,\tilde{\varphi}_{m-2}(q_{\lambda_3}\dots q_{\lambda_m})$$
$$+ 3\sum_{(\lambda_1,\lambda_2,\lambda_3)=1}^{m} h_1(q_{\lambda_1} q_{\lambda_2} q_{\lambda_3})\,\tilde{\varphi}_{m-2}(q_{\lambda_4}\dots q_{\lambda_m}; q_{\lambda_1}+q_{\lambda_2}+q_{\lambda_3})$$
$$+ \sum_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=1}^{m} h_2(q_{\lambda_1} q_{\lambda_2} q_{\lambda_3} q_{\lambda_4})\, 2\pi\,\delta(q_{\lambda_1}+q_{\lambda_2}+q_{\lambda_3}+q_{\lambda_4})$$
$$\times\, \tilde{\varphi}_{m-4}(q_{\lambda_5}\dots q_{\lambda_m}),$$

and the definitions

$$h(q) =: -2[\tilde{F}(q) + i\tilde{G}(q)]$$
$$K(q_1, q_2) =: -\operatorname*{sym}_{q_1,q_2} \tilde{G}(q_1)\,\tilde{F}(q_2) \tag{4.7}$$
$$h_1(q_1 q_2 q_3) =: -\operatorname*{sym}_{q_1 q_2 q_3} \tilde{G}(q_1)\,\tilde{F}(q_2)\,\tilde{F}(q_3)$$
$$h_2(q_1 q_2 q_3 q_4) =: -\operatorname*{sym}_{q_1 q_2 q_3 q_4} \tilde{G}(q_1)\,\tilde{F}(q_2)\,\tilde{F}(q_3)\,\tilde{F}(q_4)$$

where "sym" means symmetrization in all indices and $(\lambda_1 \dots \lambda_k)$ means the sum over all possible combinations of k elements ou tof m elements indepently of their sequence. The functions $\tilde{F}(q)$ and $\tilde{G}(q)$ are the Fourier transforms of $F(t)$ and $G(t)$ defined by (2.3) and (3.2). They are discussed in II and some features of their analytical structure in App. I. For the general solution of the system (4.5) we have to introduce according to II the "contracted" functions

$$\varphi_m^k(q_1 \dots q_k \,|\, q_{k+1} \dots q_{m-2k}) =: P_k \dots P_1\, \tilde{\varphi}_m(q_1 \dots q_m)$$
$$=: \frac{1}{(2\pi)^k} \int \tilde{\varphi}_m(q_1 - \xi_1, \xi_1 \dots q_k - \xi_k, \xi_k;$$
$$q_{k+1} \dots q_{m-2k})\, \mathrm{d}\xi_1 \dots \mathrm{d}\xi_k \qquad \left(k = 0, \dots, \left[\frac{m}{2}\right]\right) \tag{4.8}$$

with $$\mathring{\varphi}_m \equiv \tilde{\varphi}_m .$$

Since Eq. (4.5) already contains the function $\tilde{\varphi}_m^1$, we have then to derive for the functions φ_m^k an extended system of equations by applying the operators $P_k \dots P_1$ $(k = 0, \dots, [m/2])$ to the original Eqs. (4.5). This new system reads in abbreviated notation

$$\sum_{\mu,\nu} W_{m\mu}^{k\nu}\, \varphi_\mu^\nu = 0 \qquad \begin{pmatrix} m = 1, \dots, \infty \\ k = 0, \dots, \left[\frac{m}{2}\right] \end{pmatrix}. \tag{4.9}$$

The system (4.9) is then truncated in the sense of the N.T.D.-method [13] at $m = N$ i.e. we put

$$\varphi_\mu^\nu = 0 \text{ for } \mu > N;\ \nu = 0, \dots, [N/2] \tag{4.10}$$

and can then calculate φ_μ^ν for

$$\mu = 1, \dots, N, \quad \nu = 0, \dots, \left[\frac{N}{2}\right].$$

In this paper we shall investigate the case $N = 3$ numerically, while in II the solution procedure has been demonstrated at the $\varphi_2 - \varphi_4$ system explicitly without having treated the resulting eigenvalue equation numerically.

5. *The $N = 3$ Approximation*

a) Systematic solution

In the case of $N = 3$ we have to put $\varphi_m \equiv 0$ for $m \geqq 5$, and the truncated Eqs. (4.5) read (identifying $\overset{1}{\varphi}_3$ with $\bar{\varphi}_3$) [19]

$$\tilde{\varphi}_1(p) + \tilde{G}(p) \int \bar{\varphi}_3(\eta \,|\, p - \eta)\, \frac{\mathrm{d}\eta}{2\pi} = 0, \tag{5.1}$$

$$\tilde{\varphi}_3(q_1 q_2 q_3) \tag{5.2}$$
$$= \sum_{(\lambda_1,\lambda_2)=1}^{3} K(q_{\lambda_1} q_{\lambda_2})\, \bar{\varphi}_3(q_{\lambda_1}+q_{\lambda_2} \,|\, q_{\lambda_3}) + r_3(q_1 q_2 q_3)$$
$$= \sum_{l=1}^{3} K(q_i q_k)\, \bar{\varphi}_3(q_i + q_k \,|\, q_l) + r_3(q_1 q_2 q_3)$$

with

$$r_3(q_1 q_2 q_3) \tag{5.3}$$
$$=: \sum_{l=1}^{3} \tfrac{1}{3} h(q_l)\, 2\pi\,\delta(q_i + q_k)\,\tilde{\varphi}_1(q_l) + h_1(q_1 q_2 q_3).$$

Applying the contraction procedure (4.8) to (5.2) we get for the contracted function $\bar{\varphi}_3(q_1 \,|\, q_2)$ the equation

$$\bar{\varphi}_3(q_1 \,|\, q_2)\,[1 - \bar{K}(q_1)] \tag{5.4}$$
$$= 2\int K(q_1 - \alpha, q_2)\, \bar{\varphi}_3(q_1 + q_2 - \alpha, \alpha)\, \frac{\mathrm{d}\alpha}{2\pi} + \bar{r}_3(q_1 \,|\, q_2).$$

Thus we have to solve the system $\{\varphi_1, \varphi_3, \bar{\varphi}_3\}$ for $N = 3$. The Eqs. (5.1), (5.2) and (5.4) represent in this case the symbolical Eqs. (4.9). As we are only interested in the calculation of eigenvalues we can omit Eq. (5.2) for $\tilde{\varphi}_3$. We now define

$$\gamma(q) =: [1 - \bar{K}(q)]^{-1} \text{ (s.[20])} \tag{5.5}$$

and get the following equations necessarily to be fulfilled

$$\tilde{\varphi}_1(q) = -\tilde{G}(q) \int \bar{\varphi}_3(\eta \,|\, q - \eta)\, \mathrm{d}\eta / 2\pi, \tag{5.6}$$

[19] The contracted function $\bar{\varphi}_3(p_1 \,|\, p_2)$ corresponds in configurational space to the φ_3-function with two time coordinates put equal i.e. $\varphi_3(t_1 t_1 t_2)$

[20] $\bar{K}(p)$ is defined analogous to $\bar{\varphi}_3(p_1 \,|\, p_2)$ namely

$$\bar{K}(p) = \int K(p - \xi, \xi)\, \frac{\mathrm{d}\xi}{2\pi}.$$

$$\bar{\varphi}_3(q_1\,|\,q_2) = \gamma(q_1)\,[\bar{r}_3(q_1\,|\,q_2) + 2\int K(\alpha - q_2, q_2)\,\bar{\varphi}_3(\alpha, q_1 + q_2 - \alpha)\,\mathrm{d}\alpha/2\pi]\,. \tag{5.7}$$

Eq. (5.7) is an integral equation for $\bar{\varphi}_3(q_1\,|\,q_2)$ which can be brought into the standard form of an integral equation for $\bar{\varphi}_3(\eta\,|\,q-\eta)$ by the substitution

$$q_1 =: \eta; \quad q_2 =: q - \eta; \quad q_1 + q_2 = q\,, \tag{5.8}$$

$$\bar{\varphi}_3(\eta\,|\,q-\eta) = \gamma(\eta)\Big[\bar{r}_3(\eta\,|\,q-\eta) + 2\int K(\alpha - (q-\eta), q-\eta)\,\bar{\varphi}_3(\alpha\,|\,q-\alpha)\,\frac{\mathrm{d}\alpha}{2\pi}\Big]. \tag{5.9}$$

The solution of (5.9) is

$$\bar{\varphi}_3(\eta\,|\,q-\eta) = \gamma(\eta)\Big[\bar{r}_3(\eta\,|\,q-\eta) + \int R(q-\eta;\alpha)\,\bar{r}_3(\alpha\,|\,q-\alpha)\,\frac{\mathrm{d}\alpha}{2\pi}\Big] \tag{5.10}$$

with the resolvent $R(q-\eta;\alpha)$ belonging to the kernel

$$M(q-\eta;\alpha) =: 2\,\gamma(\alpha)\,K(\alpha - (q-\eta), q-\eta)\,. \tag{5.11}$$

The integral Eq. (5.9) and the construction of the resolvent R will be discussed in App. II for the approximated functions $\tilde{F}$ and $\tilde{G}$ of App. I. Here the formal existence may be assumed. Inserting now (5.10) in (5.6) we have with the definition

$$R^1(q;\eta) =: \gamma(\eta) + \int R(q-\alpha;\eta)\,\gamma(\alpha)\,\frac{\mathrm{d}\alpha}{2\pi} \tag{5.12}$$

the equation

$$\tilde{\varphi}_1(q) = -\,\tilde{G}(q)\int \bar{r}_3(\eta\,|\,q-\eta)\,R^1(q;\eta)\,\frac{\mathrm{d}\eta}{2\pi}\,. \tag{5.13}$$

Calculating $\bar{r}_3(q_1\,|\,q_2)$ from (5.3) and using the centre of gravity condition (3.11) we finally obtain after q-integration the eigenvalue equation:

$$3\,F(0) - \omega^2 + \frac{1}{3}\,A\,R^1(\omega;0) + \int\Big[\frac{2}{3}\,h(\eta) + \bar{h}_1(\omega-\eta\,|\,\eta)\Big]\,R^1(\omega;\omega-\eta)\,\frac{\mathrm{d}\eta}{2\pi} = 0 \tag{5.14}$$

with the definition

$$A =: \int h(\xi)\,\frac{\mathrm{d}\xi}{2\pi} = -\,2[F(0) + i\,G(0)]\,. \tag{5.15}$$

b) Linear combination of $\tilde{\varphi}_1$ and $\bar{\varphi}_3$

Observing the definition (4.7) of h_1 the original Eq. (5.2) may be written also

$$\tilde{\varphi}_3(q_1\,q_2\,q_3) = \sum_{l=1}^{3}\{K(q_i\,q_k)\,\tilde{\chi}(q_i + q_k;\,q_l) + \tfrac{1}{3}\,h(q_i)\,2\pi\,\delta(q_i + q_k)\,\tilde{\varphi}_1(q_l)\} \tag{5.16}$$

with the definition

$$\tilde{\chi}(q_1;\,q_2) =: \bar{\varphi}_3(q_1\,|\,q_2) + \tilde{F}(q_2)\,\tilde{\varphi}_1(q_1 + q_2) \tag{5.17}$$

reading in configurational space

$$\chi(t_1;\,t_2) = \varphi_3(t_1\,t_1\,t_2) + F(t_1 - t_2)\,\varphi_1(t_1)\,. \tag{5.18}$$

This combination has been used by DÜRR and WAGNER[3] in nonlinear spinor theory and seems to be more advantageous than the original formulation. Thus we shall use a linear combination of $\bar{\varphi}_3$ and φ_1, by which the solution of the original problem will not be altered.

From (5.16) we get in the same way as before the same integral Eq. (5.9) for $\tilde{\chi}$ as for $\bar{\varphi}_3$ with the modified inhomogeneous part

$$\bar{r}_3^{(1)}(\eta\,|\,q-\eta) = \tfrac{1}{3}\,[A\cdot 2\pi\,\delta(\eta) - 4\,i\,\tilde{G}(q-\eta) - \tilde{F}(q-\eta)]\,\tilde{\varphi}_1(q)\,. \tag{5.19}$$

This means that the solution can be constructed by the same resolvent R as in (5.16):

$$\tilde{\chi}(\eta;\,q-\eta) = \gamma(\eta)\Big[\bar{r}_3^{(1)}(\eta\,|\,q-\eta) + \int R(q-\eta;\alpha)\,\bar{r}_3^{(1)}(\alpha\,|\,q-\alpha)\,\frac{\mathrm{d}\alpha}{2\pi}\Big]. \tag{5.20}$$

Inserting (5.20) together with (5.17) in (5.6) we get analogously as before:

$$2\,F(0) - \omega^2 + \frac{1}{3}\,A\,R^1(\omega;0) - \frac{1}{3}\int[4\,i\,\tilde{G}(\eta) + \tilde{F}(\eta)]\,R^1(\eta;\omega-\eta)\,\frac{\mathrm{d}\eta}{2\pi} = 0\,. \tag{5.21}$$

c) Iterated eigenvalue equation

As the integral equations for $\bar{\varphi}_3$ resp. $\tilde{\chi}$ cannot be solved exactly and approximations have to be made we shall use for our calculations an iterated one instead of the system (5.6) and (5.7). From the rearranged Eq. (5.16) we then obtain the very simple expression

$$\int\bar{\varphi}_3(\eta\,|\,q-\eta)\,\frac{\mathrm{d}\eta}{2\pi} = A\cdot\tilde{\varphi}_1(q) + 3\int\bar{K}(\alpha)\,\tilde{\chi}(\alpha;\,q-\alpha)\,\frac{\mathrm{d}\alpha}{2\pi}\,. \tag{5.22}$$

This inserted in (5.6) results in the new iterated equation

$$\tilde{\varphi}_1(q)\,[\tilde{G}^{-1}(q) + A] + 3\int\bar{K}(\alpha)\,\tilde{\chi}(\alpha;\,q-\alpha)\,\frac{\mathrm{d}\alpha}{2\pi} = 0\,. \tag{5.23}$$

Now, with the solution (5.20) of the integral eqaution for $\tilde{\chi}$ we get from (5.23) the eigenvalue equation

for ω which we shall investigate numerically

$$3F(0) - \omega^2 + A[1 + R^2(\omega;0)] - \int [\tilde{F}(\eta) + 4i\tilde{G}(\eta)]\, R^2(\omega;\omega-\eta)\frac{d\eta}{2\pi} = 0 \tag{5.24}$$

with the definition

$$R^2(\omega;\eta) =: \gamma(\eta)\bar{K}(\eta) + \int \gamma(\xi)\bar{K}(\xi)\, R(\omega-\xi;\eta)\frac{d\xi}{2\pi}. \tag{5.25}$$

This equation differs slightly from (5.21) although it already incorporates one additional iteration step.

6. *Approximated Solutions*

a) The singular functions $\tilde{F}_{app}(p)$ and $\tilde{G}_{app}(p)$

For further investigations we use the approximated two-point function $\tilde{F}_{app}(p)$ and Green's function $\tilde{G}_{app}(p)$ defined in II App. V which read

$$\tilde{F}_{app}(p) \equiv i f(p) =: i[p^2 - \omega_1^2 + i\varepsilon]^{-1}, \tag{6.1}$$

$$\tilde{G}_{app}(p) \equiv: -g(p) =: -[p^2 - a^2 + i\varepsilon]^{-1} \tag{6.2}$$

with $a^2 = 3/2\,\omega_1$ and where the small imaginary part indicates Feynman integration. In the approximate version (6.1) we have to consider ω_1 as a parameter which still has to be determined. There are two possibilities to fix the parameter ω_1. Either we take it from other, already known calculations, as

α) the simplest approximation of the system for vacuum expectation values,

β) a simpler problem, already known, — here the harmonic oscillator —,

γ) the exact value given by SCHRÖDINGER theory[10].

Or we calculate ω_1 selfconsistently from the various eigenvalue equations by fixing ω_1 in such a way, that it coincides with the lowest eigenvalue obtained. The second possibility is the most appropriate one, since in this case only the eigenvalues are calculated without any further information. Moreover we have then the full analogy to calculations in nonlinear spinor theory. Both possibilities will be compared in our calculations. In the first case, we have the following ω_1-values:

$$\alpha)\ 1.1447,\quad \beta)\ 1.0000,\quad \gamma)\ 1.0871. \tag{6.3}$$

b) The eigenvalue Eq. (5.24)

With these approximated singular functions we get at once according to (I.4)

$$\bar{K}(p) = \frac{a+\omega_1}{a\cdot\omega_1}[p^2 - (a+\omega_1)^2 + i\varepsilon]^{-1} \tag{6.4}$$

and from (5.5) we get

$$\gamma(p) = \frac{p^2 - (a+\omega_1)^2}{p^2 - \gamma^2 + i\varepsilon} \tag{6.5}$$

with $$\gamma^2 =: (a+\omega_1)^2 + \frac{a+\omega_1}{a\cdot\omega_1}.$$

Definition (5.15) now becomes

$$A = (\omega_1 - a)/(\omega_1 \cdot a) \tag{6.6}$$

and using (5.25) as well as the auxiliary formula (I.3) the approximated eigenvalue Eq. (5.24) has the structure:

$$\begin{aligned} &\frac{\omega_1^2 - a^2}{\gamma^2} + \frac{a\cdot\omega_1}{(a+\omega_1)}[a^2 - \omega^2] \\ &\quad + 2\frac{(a+\gamma)}{a\gamma}[\omega^2 - (a+\gamma)^2]^{-1} \\ &\quad - \frac{(\omega_1+\gamma)}{2\omega_1\gamma}[\omega^2 - (\omega_1+\gamma)^2]^{-1} + AN(\omega;0) \\ &\quad + \frac{(-)}{2\pi i}\int [4g(\eta) - f(\eta)]\, N(\omega;\omega-\eta)\, d\eta \\ &\quad \equiv W(\omega;\omega_1) = 0 \end{aligned} \tag{6.7}$$

with the definition

$$N(\omega;\eta) =: \int \frac{R(\xi;\eta)}{[(\xi-\omega)^2 - \gamma^2 + i\varepsilon]}\frac{d\xi}{2\pi}. \tag{6.8}$$

Now, according to App. II we represent the resolvent $R(\omega-\xi;\eta)$ of the integral Eq. (5.8) by a Neumann-series, the first term of which we will then use. Thus, we replace R by the kernel (5.11) itself and (6.8) reads

$$N^{(1)}(\omega;\eta) = 2\gamma(\eta)\frac{(-)}{2\pi i}\int \frac{[f(\xi)g(\eta-\xi) + g(\xi)f(\eta-\xi)]}{[(\xi-\omega)^2 - \gamma^2 + i\varepsilon]}\, d\xi. \tag{6.9}$$

The calculation of (6.9) is carried out by closing the integration contour into the negative imaginary half plane and using the residual theorem. This is done in App. I. The eigenvalue equation in this approximation is then also given there because the expressions get somewhat lengthy. We get $W(\omega;\omega_1)$ to be a meromorphic function of ω and ω_1.

Other approximation possibilities will not be considered, i.e. especially a systematic investigation of the first terms of the Neumann series.

c) Simple approximations

α) $N = 1$.

The most simple approximation is for $N = 1$. Then we put $\tilde{\varphi}_3(q_1q_2q_3) \equiv 0$ and get at once from (5.16)

$$\tilde{G}_{app}^{-1}(\omega) \equiv a^2 - \omega^2 = 0. \tag{6.10}$$

This gives because of $a^2 = 3/2\,\omega_1$ the selfconsistent eigenvalue[21]

$$\omega = \omega_1 = \sqrt[3]{\frac{3}{2}} \approx 1.1447\,. \tag{6.11}$$

This eigenvalue was already obtained by HEISENBERG[7] and KAISER[9] from the lowest one-time approximations. Moreover we get this value in the simplest approximation of the twopoint function $F(t_1 - t_2)$. It is remarkable that, at this value, we have $g(p) \equiv f(p)$ because of $\omega_1 = a$, i.e. the approximated Green's function and twopoint function are identical. By this, Eq. (5.2) becomes much simpler, because additionally we have $h(q) \equiv 0$.

β) Lowest $N = 3$ approximation

In the simplest approximation we consider only φ_1-contributions to determine $\tilde{\chi}(p_1; p_2)$. Thus we put in (5.17) $\bar{\varphi}_3 \equiv 0$ and use in Eq. (5.23)

$$\tilde{\chi}_0(p_1; p_2) =: \tilde{F}(p_2)\,\tilde{\varphi}_1(p_1 + p_2) \tag{6.12}$$

with the result according to (6.1) and (6.2) (6.13)

$$\left[A + a^2 - p^2 + 3\,\frac{(-)}{2\pi i}\int \bar{K}(\alpha) f(p-\alpha)\,\mathrm{d}\alpha\right]\tilde{\varphi}_1(p) = 0\,.$$

Using (6.4) and (I.3) we have

$$\frac{(-)}{2\pi i}\int \bar{K}(\alpha) f(p-\alpha)\,\mathrm{d}\alpha = \frac{(a+2\omega_1)}{2a\,\omega_1}\,[p^2 - (a+2\omega_1)^2 + i\varepsilon]^{-1} \tag{6.14}$$

and with (3.11) Eq. (6.13) results after p-integration in the eigenvalue equation

$$W^0(\omega; \omega_1) \equiv A + a^2 - \omega^2 + 3\,\frac{(a+2\omega_1)}{2a\,\omega_1^2}\,[\omega^2 - (a+2\omega_1)^2]^{-1} = 0\,. \tag{6.15}$$

III. The Hermitean Representation

7. *The ϱ-equations*

Now we consider the other possibility, namely to represent $\mathscr{T}_\varrho(j)$ by Hermite base functionals

$$J_n(t_1 \dots t_n; j)\,.$$

Again we start with the Dyson expansion (3.3). However, for the evaluation of the symmetrical matrix elements it is necessary to perform genuine functional integration between different base functionals (3.1). This is done by a transformation of the base functionals into the standard form by changing the source functions $j(t)$. Substituting the transformation

$$j(t) = \int K(t - t')\,h(t')\,\mathrm{d}t' \tag{7.1}$$

in (3.1) and (3.3), where $K(t-t')$ is defined by

$$\int K(t_1 - t)\,F(t - t')\,K(t' - t_2)\,\mathrm{d}t\,\mathrm{d}t' = \tfrac{1}{2}\,\delta(t_1 - t_2) \tag{7.2}$$

we obtain the expansion

$$\mathscr{T}_\varrho(h) = \sum_{n=1}^{\infty} \int \chi_n(\xi_1 \dots \xi_n)\,\mathrm{d}_n(\xi_1 \dots \xi_n; h)\,\mathrm{d}\xi_1 \dots \mathrm{d}\xi_n \tag{7.3}$$

with the standard Dyson functionals

$$\mathrm{d}_n(t_1 \dots t_n; h) =: \frac{i^n}{n!}\,h(t_1) \dots h(t_n) \exp\left[-\frac{1}{4}\int h^2(\xi)\,\mathrm{d}\xi\right] \tag{7.4}$$

and

$$\chi_n(t_1 \dots t_n) =: \int K(t_1 - \xi_1) \dots K(t_n - \xi_n)\,\varphi_n(\xi_1 \dots \xi_n)\,\mathrm{d}\xi_1 \dots \mathrm{d}\xi_n\,. \tag{7.5}$$

By transformation of (7.2) into Fourier space, we can determine $K(t_1 - t_2)$ to

$$K(t_1 - t_2) = \frac{1}{2\pi}\int \tilde{K}(p)\,e^{-i(t_1 - t_2)p}\,\mathrm{d}p = \frac{1}{2\pi}\int \frac{e^{-i(t_1 - t_2)p}}{\sqrt{2\,\tilde{F}(p)}}\,\mathrm{d}p\,. \tag{7.6}$$

But we shall see later, that all $K(t_1 - t_2)$ can be eliminated in the final equations.

Having transformed the state functionals we have to do the same procedure with the functional operators. Therefore we use the relation, resulting from (7.1)

$$\frac{\delta}{\delta j(t)} = \int K^{-1}(t - t')\,\frac{\delta}{\delta h(t')}\,\mathrm{d}t' \tag{7.7}$$

and choose the smearing out operator $\mathbf{S}$ of (2.5) to be

$$\mathbf{S} =: \int\left[\frac{\delta}{\delta j(\xi)}\,K(\xi - t') - \frac{1}{2}\,j(\xi)\,K^{-1}(\xi - t')\right] \times s(t' - t)\,\mathrm{d}\xi\,\mathrm{d}t\,\mathrm{d}t' \tag{7.8}$$

where $s(t - t')$ is a still undefined function which is chosen only for practical aspects of integrability. Then the functional Eq. (2.6) reads

$$-\int A^+(\xi)\,\hat{\mathbf{0}}\left(h; \frac{\delta}{\delta h}; \xi\right)\mathrm{d}\xi\,\mathscr{T}_\varrho(h) = 0 \tag{7.9}$$

with the "creation" operator A^+ defined by

$$A^+(t) =: \tfrac{1}{2}\,h(t) - \delta/\delta h(t)\,, \tag{7.10}$$

[21] We shall get the same result, if we put in (5.23) $\bar{\chi}(p_1; p_2) \equiv 0$.

and

$$\hat{\mathbf{0}}\left(h, \frac{\delta}{\delta h}; \xi\right) =: \int s(\xi - t)\, \mathbf{0}\left(\int K(t-\eta)\, h(\eta)\, \mathrm{d}\eta\,, \cdot \int K^{-1}(t-\eta)\, \frac{\delta}{\delta h(\eta)}\, \mathrm{d}\eta\right) \mathrm{d}t. \quad (7.11)$$

It must be noted, that the Wick ordering is different in the unsymmetrical Dyson- and the symmetrical Hermite representation, because it is characterized in functional notation by the corresponding exponential functionals[18 b]

$$\exp[-\tfrac{1}{2}\int j(\xi)\, F(\xi-\eta)\, j(\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta] \quad \text{resp.}$$
$$\exp[-\tfrac{1}{4}\int h^2(\xi)\, \mathrm{d}\xi].$$

By this the undefined factor $\varkappa$ in the Green's function (2.3) is given by

$$\varkappa = \begin{matrix} 1 \text{ Dyson unsymmetrical} \\ \tfrac{1}{2} \text{ Hermitean symmetrical.} \end{matrix} \quad (7.12)$$

Instead of the standard Dyson functionals

$$\mathrm{d}_n(t_1 \dots t_n; h),$$

which are not orthogonal in respect of Hilbert space integration, we now use orthogonal Hermite functionals $J_n(t_1 \dots t_n; h)$ defined in III which satisfy (3.5). With the help of the representation of the unit operator in functional space

$$\delta(h, h') \equiv \sum_{n=1}^{\infty} \int |J_n(t_1 \dots t_n; h)\rangle \langle J_n(t_1 \dots t_n; h')|\, \mathrm{d}t_1 \dots \mathrm{d}t_n \quad (7.13)$$

the matrix representation of the functional Eq. (7.9) now reads

$$-\langle J_k(t_1 \dots t_k)| \int A^+(\xi)\, \hat{\mathbf{0}}\left(h, \frac{\delta}{\delta h}; \xi\right) \mathrm{d}\xi\, |\mathscr{T}_\varrho(h)\rangle \sqrt{k!} = \quad (7.14)$$
$$-\sum_{m=1}^{\infty} \frac{\sqrt{k!}}{\sqrt{m!}} \int \langle J_k(t_1 \dots t_k)|\, A^+(\xi)\, |J_m(x_1 \dots x_m)\rangle f_m(x_1 \dots x_m; \xi)\, \mathrm{d}\xi\, \mathrm{d}x_1 \dots \mathrm{d}x_m = \sum_{\lambda_1=1}^{m} f_{m-1}(t_{\lambda_2} \dots t_{\lambda_m}; t_{\lambda_1}) = 0$$

with
$$f_m(x_1 \dots x_m; \xi) =: \langle J_m(x_1 \dots x_m) \left| \hat{\mathbf{0}}\left(h, \frac{\delta}{\delta h}; \xi\right) \right| \mathscr{T}_\varrho(h)\rangle \sqrt{m!}. \quad (7.15)$$

Thus the operator S in (7.8) is shown to symmetrize the individual equations $f_m(x_1 \dots x_m; \xi) = 0$. By means of the recurrence relations for the operators h and $\delta/\delta h$ we can calculate the matrix elements (7.15) analogously as in III and get — already transformed in Fourier space —

$$\begin{aligned} \tilde{f}_k(p_1 \dots p_k; p) \equiv \tilde{s}(p)\, \tilde{G}(p) \Big\{ & \frac{1}{8} \int \varrho^1_{k+3}(p - \xi\,|\, \xi\, p_1 \dots p_k) \frac{\mathrm{d}\xi}{2\pi} \\ & + \frac{3}{4} \sum_{\lambda_1=1}^{k} \tilde{F}(p_{\lambda_1})\, \varrho^1_{k+1}(p + p_{\lambda_1}|\, p_{\lambda_2} \dots p_{\lambda_k}) + \frac{1}{2} B^+(p)\, \tilde{\varrho}_{k+1}(p_1 \dots p_k; p) \\ & + 3 \sum_{(\lambda_1, \lambda_2)=1}^{k} \tilde{F}(p_{\lambda_1})\, \tilde{F}(p_{\lambda_2})\, \tilde{\varrho}_{k-1}(p_{\lambda_3} \dots p_{\lambda_k}; p_{\lambda_2} + p_{\lambda_3} + p) + \sum_{\lambda_1=1}^{k} B^-(p)\, \tilde{F}(p)\, 2\pi\, \delta(p + p_{\lambda_1})\, \tilde{\varrho}_{k-1}(p_{\lambda_2} \dots p_{\lambda_k}) \\ & + 6 \sum_{(\lambda_1, \lambda_2, \lambda_3)=1}^{k} \tilde{F}(p_{\lambda_1})\, \tilde{F}(p_{\lambda_2})\, \tilde{F}(p_{\lambda_3})\, 2\pi\, \delta(p + p_{\lambda_1} + p_{\lambda_2} + p_{\lambda_3})\, \tilde{\varrho}_{k-3}(p_{\lambda_4} \dots p_{\lambda_k}) \Big\} \end{aligned} \quad (7.16)$$

with the definition
$$B^{\pm}(p) =: \tilde{G}^{-1}(p) \pm \frac{1}{i} \tilde{F}^{-1}(p) \quad (7.17)$$

and the transformed expansion functions

$$\tilde{\varrho}_j(p_1 \dots p_j) =: \tilde{K}(p_1) \dots \tilde{K}(p_j)\, \tilde{\eta}_j(p_1 \dots p_j)\, i^j. \quad (7.18)$$

The contracted functions are defined as in (4.8). From (7.16) a most appropriate smearing out function $\tilde{s}(p)$ can easily be derived namely

$$\tilde{s}(p) =: \tilde{G}^{-1}(p)\, 2\, B^+(p)^{-1}. \quad (7.19)$$

Then we get after symmetrization the most favourable diagonal term $k \cdot \tilde{\varrho}_k(p_1 \dots p_k)$ whereas every $\tilde{s}(p)$ differing from (7.19) would result in a diagonal term of the kind

$$\left[\tfrac{1}{2} \sum_{j=1}^{k} \tilde{s}(p_j)\, B^+(p_j)\right] \tilde{\varrho}_k(p_1 \dots p_k).$$

With this special $\tilde{s}(p)$ the final symmetrized equation system has the same structure as in the Dyson representation (4.5). They only differ in the coefficient functions. The general form of the equation system for both representations is

$$\begin{aligned} k\cdot\tilde{\varrho}_k(p_1\dots p_k) &= \sum_{\lambda_1=1}^{k} h_0(p_{\lambda_1})\int \varrho^1_{k+1}(p_{\lambda_1}-\xi\,|\,\xi\,p_{\lambda_2}\dots p_{\lambda_k})\frac{\mathrm{d}\xi}{2\pi} + 3\sum_{(\lambda_1,\lambda_2)=1}^{k} K(p_{\lambda'_1}p_{\lambda_2})\,\varrho^1_k(p_{\lambda_1}+p_{\lambda_2}\,|\,p_{\lambda_3}\dots p_{\lambda_k}) \\ &+ 3\sum_{(\lambda_1,\lambda_2,\lambda_3)=1}^{k} h_1(p_{\lambda_1}p_{\lambda_2}p_{\lambda_3})\,\tilde{\varrho}_{k-2}(p_{\lambda_4}\dots p_{\lambda_k};\,p_{\lambda_1}+p_{\lambda_2}+p_{\lambda_3}) + \sum_{(\lambda_1,\lambda_2)=1}^{k} h(p_{\lambda_1})\,2\pi\,\delta(p_{\lambda_1}+p_{\lambda_2})\,\tilde{\varrho}_{k-2}(p_{\lambda_3}\dots p_{\lambda_k}) \\ &+ \sum_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=1}^{k} h_2(p_{\lambda_1}p_{\lambda_2}p_{\lambda_3}p_{\lambda_4})\,2\pi\,\delta(p_{\lambda_1}+p_{\lambda_2}+p_{\lambda_3}+p_{\lambda_4})\,\tilde{\varrho}_{k-4}(p_{\lambda_5}\dots p_{\lambda_k})\,. \end{aligned} \tag{7.20}$$

The coefficient functions are defined by

	Hermite	Dyson representation
$h_0(p)$	$\frac{1}{8}B(p)$	$\tilde{G}'(p)$
$h(p)$	$2\left[\tilde{G}^{-1}(p)-\frac{1}{i}\tilde{F}^{-1}(p)\right]B(p)\,\tilde{F}(p)$	$-2\left[\tilde{F}(p)+\frac{1}{i}\tilde{G}'(p)\right]$
$K(p_1,p_2)$	$\frac{1}{4}\operatorname*{sym}_{p_1,p_2}\tilde{F}(p_1)\,B(p_2)$	$\operatorname*{sym}_{p_1,p_2}\tilde{F}(p_1)\,\tilde{G}'(p_2)$
$h_1(p_1p_2p_3)$	$\frac{1}{2}\operatorname*{sym}_{p_1,p_2,p_3}\tilde{F}(p_1)\,\tilde{F}(p_2)\,B(p_3)$	$\operatorname*{sym}_{p_1,p_2,p_3}\tilde{F}(p_1)\,\tilde{F}(p_2)\,\tilde{G}'(p_3)$
$h_2(p_1p_2p_3p_4)$	$\operatorname*{sym}_{p_1,p_2,p_3,p_4}\tilde{F}(p_1)\,\tilde{F}(p_2)\,\tilde{F}(p_3)\,B(p_4)$	$\operatorname*{sym}_{p_1,p_2,p_3,p_4}\tilde{F}(p_1)\,\tilde{F}(p_2)\,\tilde{F}(p_3)\,\tilde{G}'(p_4)$
$B(p)=:2\left[\tilde{G}^{-1}(p)+\frac{1}{i}\tilde{F}(p)^{-1}\right]$		$\tilde{G}'(p)=:-\tilde{G}(p)=[p^2-\varkappa 3F(0)+i\varepsilon]^{-1}$.

(7.21)

Thus, disregarding numerical factors, the only difference between both representations is the substitution of $\tilde{G}'(p)$ by $B(p)$ with the exception of $h(p)$. Therefore the integration procedure of chapter 4 is also valid for both of them. For identification we mark the Hermitean coefficient functions by an index H.

8. *The $N=3$ Approximation*

a) The linear combined, iterated eigenvalue equation

Truncating the system (7.20) for $N=3$ i.e. putting $\varrho_k\equiv 0$ for $k>3$ we get analogously to equations (5.1)–(5.3) in the Dyson case

$$\tilde{\varrho}_1(q)=\frac{1}{8}B(q)\int\bar{\varrho}_3(\eta\,|\,q-\eta)\frac{\mathrm{d}\eta}{2\pi}\,, \tag{8.1}$$

$$\begin{aligned} &\tilde{\varrho}_3(q_1q_2q_3) \\ &\quad=\sum_{l=1}^{3}K_{\mathrm{H}}(q_iq_k)\,\bar{\varrho}_3(q_i+q_k\,|\,q_l)+r_3^{\mathrm{H}}(q_1q_2q_3) \end{aligned} \tag{8.2}$$

with

$$\begin{aligned} r_3^{\mathrm{H}}(q_1q_2q_3) &=:\sum_{l=1}^{3}\tfrac{1}{3}h_{\mathrm{H}}(q_i)\,2\pi\,\delta(q_i+q_k)\,\tilde{\varrho}_1(q_l) \\ &\quad+h_1^{\mathrm{H}}(q_1q_2q_3)\,\tilde{\varrho}_1(q_1+q_2+q_3) \end{aligned} \tag{8.3}$$

and

$$\begin{aligned} &\bar{\varrho}_3(q_1\,|\,q_2) \\ &\quad=:\varrho_3^1(q_1\,|\,q_2)=\frac{1}{2\pi}\int\varrho_3(q_1-\xi,\xi,q_2)\,\mathrm{d}\xi\,. \end{aligned} \tag{8.4}$$

Like in the Dyson case we use the iterated, linearly combined eigenvalue equation for numerical calculations. All calculational steps run completely analogous to those in Sect. 5, and the resulting eigenvalue equations show the same structure. Therefore the other possibilities will not be discussed here once more. Again we introduce a linearly combined function

$$\tilde{\chi}_{\mathrm{H}}(q_1;q_2)=:\bar{\varrho}_3(q_1\,|\,q_2)+2\tilde{F}(q_2)\,\tilde{\varrho}_1(q_1+q_2) \tag{8.5}$$

and Eq. (8.1) iterated once with (8.2) reads then analogous to (5.23)

$$\begin{aligned} &\tilde{\varrho}_1(q)\,[A_{\mathrm{H}}-8\,B^{-1}(q)] \\ &\quad+3\int\bar{K}_{\mathrm{H}}(\xi)\,\tilde{\chi}_{\mathrm{H}}(\xi;q-\xi)\,\mathrm{d}\xi/2\pi=0 \end{aligned} \tag{8.6}$$

with the definition

$$A_{\mathrm{H}}=:\int h_{\mathrm{H}}(\xi)\,\mathrm{d}\xi/2\pi\,. \tag{8.7}$$

The integral equation for $\tilde{\chi}_{\mathrm{H}}(q_1;q_2)$ derived from (8.2) has the same structure as (5.9):

$$\begin{aligned} &\tilde{\chi}_{\mathrm{H}}(\eta;q-\eta)=\gamma_{\mathrm{H}}(\eta)\,[W_{\mathrm{H}}(\eta;q-\eta) \\ &\quad+2\int K_{\mathrm{H}}(q-\eta,\xi-(q-\eta))\,\tilde{\chi}_{\mathrm{H}}(\xi;q-\xi)\,\mathrm{d}\xi/2\pi] \end{aligned} \tag{8.8}$$

with

$$W_H(\eta; q-\eta) \tag{8.9}$$
$$=: [\tfrac{1}{3} A_H \cdot 2\pi\delta(\eta) + \tfrac{1}{2} h_H(q-\eta) + 2\tilde{F}(q-\eta)]\,\tilde{\varrho}_1(q)$$

and

$$\gamma_H(\eta) =: [1 - \bar{K}_H(\eta)]^{-1}. \tag{8.10}$$

The solution of (8.8) is analogous to (5.20)

$$\tilde{\chi}_H(\eta; q-\eta) = \gamma_H(\eta)\,[W_H(\eta, q-\eta) + \int R_H(q-\eta;\xi)\, W_H(\xi, q-\xi)\,\mathrm{d}\xi/2\pi] \tag{8.11}$$

with the resolvent $R_H(q-\eta;\xi)$ belonging to the kernel (8.12)

$$M_H(q-\eta;\xi) =: 2\gamma_H(\xi)\, K_H(q-\eta, \xi-(q-\eta))$$

Inserting (8.11) and (8.9) in (8.6) we get, because of $\tilde{\varrho}_1(p) = c_0 \cdot \delta(p-\omega)$ after p-integration the final eigenvalue equation

$$\tilde{G}_H^{-1}(\omega) + \frac{1}{i}\tilde{F}^{-1}(\omega) + \frac{1}{4} A_H[1 + R_H^2(\omega;0)] \tag{8.13}$$
$$+ \frac{3}{2}\int\left[\frac{1}{3} h_H(\xi) + \tilde{F}(\xi)\right] R_H^2(\omega;\omega-\xi)\,\frac{\mathrm{d}\xi}{2\pi} = 0$$

with the definition

$$R_H^2(\omega;\xi) =: \gamma_H(\xi)\,\bar{K}_H(\xi) + \int \gamma_H(\eta)\,\bar{K}_H(\eta)\, R_H(\omega-\eta;\xi)\,\frac{\mathrm{d}\eta}{2\pi}. \tag{8.14}$$

b) The approximated solution of Eq. (8.13)

Using again the approximative functions $\tilde{F}\mathrm{app}(p)$ and $\tilde{G}\mathrm{app}(p)$ but now with $a^2 = 3/4\omega_1$ we have from (7.21)

$$B(q) =: b(q) = [q^2 - b^2 + i\varepsilon]^{-1} \tag{8.15}$$

with $$b^2 = \tfrac{1}{2}[a^2 + \omega_1^2]$$

and because of (I.4)

$$\bar{K}_H(p) = \frac{b+\omega_1}{4b\omega_1}[p^2 - (b+\omega_1)^2 + i\varepsilon]^{-1}. \tag{8.16}$$

Then (8.10) becomes

$$\gamma_H(p) = \frac{p^2 - (b+\omega_1)^2}{p^2 - \gamma_H^2 + i\varepsilon} \tag{8.17}$$

with $$\gamma_H^2 =: (b+\omega_1)^2 + \frac{(b+\omega_1)}{4b\omega_1}.$$

Furthermore because of

$$h_H(p) = -2i[\omega_1^2 - a^2]\,b(p)\,f(p) = 4i[b(p) - f(p)] \tag{8.18}$$

(8.7) reads

$$A_H = \frac{2(\omega_1 - b)}{b\cdot\omega_1} \tag{8.19}$$

and (8.14) becomes

$$R_H^2(\omega;\xi) = \frac{(b+\omega_1)}{4b\omega_1} \times\left[\frac{1}{\xi^2 - \gamma_H^2 + i\varepsilon} + \int \frac{R_H(\eta;\xi)}{[(\eta-\omega)^2 - \gamma_H^2 + i\varepsilon]}\,\frac{\mathrm{d}\eta}{2\pi}\right]. \tag{8.20}$$

Now the approximative eigenvalue Eq. (8.13) can be written

$$\frac{8b\omega_1}{(b+\omega_1)}(b^2-\omega^2) + \frac{2}{\gamma_H^2}(\omega_1^2 - b^2) + \frac{(\gamma_H+b)}{\gamma_H\cdot b}[\omega^2 - (\gamma_H+b)^2]^{-1} \tag{8.21}$$
$$- \frac{(\gamma_H+\omega_1)}{4\gamma_H\cdot\omega_1}[\omega^2 - (\gamma_H+\omega_1)^2]^{-1} + \frac{1}{4} A_H N_H(\omega;0) + \frac{(-)}{2\pi i}\int\left[b(\xi) - \frac{1}{4} f(\xi)\right] N_H(\omega;\omega-\xi)\,\mathrm{d}\xi$$
$$\equiv W_H(\omega;\omega_1) = 0$$

with the definition analogue to (6.8) (8.22)

$$N_H(\omega;\xi) =: 2\cdot\frac{1}{2\pi}\int \frac{R_H(\eta;\xi)}{[(\eta-\omega)^2 - \gamma_H^2 + i\varepsilon]}\,\mathrm{d}\eta.$$

Replacing again as in the Dyson case the resolvent $R_H(\omega-\eta;\xi)$ of integral equation (8.8) by the first term of its Neumann series i.e. by the kernel (8.12) we have

$$N_H^{(1)}(\omega;\xi) = \gamma_H(\xi)\cdot\frac{(-)}{2\pi i}\int\frac{[f(\eta)\,b(\xi-\eta) + b(\eta)\,f(\xi-\eta)]}{[(\eta-\omega)^2 + \gamma_H^2 + i\varepsilon]}\,\mathrm{d}\eta. \tag{8.23}$$

The investigation of (8.23) and the resulting eigenvalue Eq. (8.21) is given again in App. I.

c) Other simple approximations

α) $N = 1$.

We now consider the same possibilities as in section 6.c of the Dyson representation. Therefore we put in Eq. (8.1) $\bar{\varrho}_3(p_1|p_2) \equiv 0$ and get

$$b^{-1}(\omega) \equiv \omega^2 - b^2 = 0. \tag{8.24}$$

Because of $b^2 = \frac{1}{2}(\omega_1^2 + a^2)$ and $a^2 = 3/4\omega_1$ this yields the selfconsistent eigenvalue[22]

$$\omega = \omega_1 = \sqrt[3]{\frac{3}{4}} \approx 0.9085. \tag{8.25}$$

This value was already obtained by MAISON[16] in the nonsymmetrized q-case and in III for the symmetrized p-q-case. For this special value we have $\omega_1 = a = b$ and therefore $g(p) \equiv f(p) \equiv b(p)$ i.e. the coefficient functions (7.21) and Eq. (8.2) become much simpler

[22] One obtains the same result if we put in (8.6) $\tilde{\chi}_H(p_1;p_2) \equiv 0$.

β) Lowest $N = 3$ approximation

Again we only use $\tilde{\varrho}_1$-contributions for $\tilde{\chi}_H(p_1; p_2)$ and put thus $\tilde{\varrho}_3 \equiv 0$. Then $\tilde{\chi}_H$ reads

$$\tilde{\chi}^0_H(p_1; p_2) = 2\tilde{F}(p_2)\,\tilde{\varrho}_1(p_1 + p_2) \tag{8.26}$$

and we get from (8.6) with (8.15)

$$\tilde{\varrho}_1(p) \tag{8.27}$$
$$\times\left[A_H + 8(b^2 - p^2) + 6\,\frac{(-)}{2\pi i}\int f(p-\xi)\,\bar{K}_H(\xi)\,\mathrm{d}\xi\right] = 0.$$

This results in the eigenvalue equation

$$W^0_H(\omega; \omega_1) \equiv A_H + 8(b^2 - \omega^2) \tag{8.28}$$
$$+ \frac{3}{4}\,\frac{(b + 2\omega_1)}{b\,\omega_1^2}\,[\omega^2 - (b + 2\omega_1)^2]^{-1} = 0.$$

IV. Other Possibilities

9. *The system of nonsymmetrized equations*

Instead of smearing out or combining linearly the original functional Eq. (2.4) with the operator S of (2.5) one can naturally investigate it directly, too. Then, as well, the solutions may not depend on the arbitrary parameter t. We have to take into consideration that the expansion functionals

$$D_n(t_1 \dots t_n; j) \quad \text{resp.} \quad J_n(t_1 \dots t_n; j)$$

for the solution functional $\mathscr{T}_\varrho(j)$ are symmetrical in all variables, wherefrom the symmetry of the expansion functions $\varphi_n(t_1 \dots t_n)$ resp. $\eta_n(t_1 \dots t_n)$ and $\varrho_n(t_1 \dots t_n)$ can be seen by (3.3) and (3.6). This means the symmetry of the expansion functions φ_n and ϱ_n in all variables represents an essential subsidiary condition to the functional Eq. (2.4). By the application of the operator S this condition has been automatically fulfilled as can be recognized from the structure of (7.20). To satisfy this condition without linear combination we have to investigate the system of matrix representations in the Dyson case

$$\langle D^m(t_1 \dots t_{k-1} t_{k+1} \dots t_{m+1}) \left| \mathbf{0}\left(j(t_k), \frac{\delta}{\delta j(t_k)}\right) \right| \mathscr{T}_\varrho(j)\rangle \sqrt{m!} = 0 \qquad \begin{pmatrix} k = 1 \dots m+1 \\ m = 1 \dots \infty \end{pmatrix} \tag{9.1}$$

respectively in the Hermite case

$$\langle J^m(t_1 \dots t_{k-1} t_{k+1} \dots t_{m+1}) \left| \mathbf{0}\left(h, \frac{\delta}{\delta h}; t_k\right) \right| \mathscr{T}_\varrho(h)\rangle \sqrt{m!} = 0. \qquad \begin{pmatrix} k = 1 \dots m+1 \\ m = 1 \dots \infty \end{pmatrix}. \tag{9.2}$$

Both equation systems have the same structure and read analogous to chapter II and III in Fourier space

$$\tilde{\varrho}_k(p_1 \dots p_k) = h_0(p_{\lambda_1}) \int \varrho^1_{k+2}(p_{\lambda_1} - \xi | \xi\, p_{\lambda_2} \dots p_{\lambda_k})\,\frac{\mathrm{d}\xi}{2\pi} + 3\sum_{\lambda_2=1}^{k} K^u(p_{\lambda_1} p_{\lambda_2})\,\varrho^1_k(p_{\lambda_1} + p_{\lambda_2} | p_{\lambda_3} \dots p_{\lambda_k}) \tag{9.3}$$
$$+ 3\sum_{(\lambda_2, \lambda_3)=1}^{k} h^u_1(p_{\lambda_1} p_{\lambda_2} p_{\lambda_3})\,\tilde{\varrho}_{k-2}(p_{\lambda_4} \dots p_{\lambda_k}; p_{\lambda_1} + p_{\lambda_2} + p_{\lambda_3}) + \sum_{\lambda_2=1}^{k} h^u(p_{\lambda_1})\,2\pi\,\delta(p_{\lambda_1} + p_{\lambda_2})\,\tilde{\varrho}_{k-2}(p_{\lambda_3} \dots p_{\lambda_k})$$
$$+ \sum_{(\lambda_2, \lambda_3, \lambda_4)=1}^{k} h^u_2(p_{\lambda_1} p_{\lambda_2} p_{\lambda_3} p_{\lambda_4})\,2\pi\,\delta(p_{\lambda_1} + p_{\lambda_2} + p_{\lambda_3} + p_{\lambda_4})\,\tilde{\varrho}_{k-4}(p_{\lambda_5} \dots p_{\lambda_k}) \qquad \begin{pmatrix} \lambda_1 = 1, \dots, k \\ k = 1, \dots, \infty \end{pmatrix}$$

with the coefficient functions:

	Hermite	Dyson
$h_0(p)$	$\frac{1}{8}\,B(p)$	$\tilde{G}'(p)$
$h^u(p)$	$\left[\tilde{G}^{-1}(p) - \frac{1}{i}\,\tilde{F}^{-1}(p)\right] B(p)\,\tilde{F}(p)$	$-\left[\tilde{F}(p) + \frac{1}{i}\,\tilde{G}'(p)\right]$
$K^u(p, p_1)$	$\frac{1}{4}\,B(p)\,\tilde{F}(p_1)$	$\tilde{G}'(p)\,\tilde{F}(p_1)$
$h^u_1(p\,p_1\,p_2)$	$\frac{1}{2}\,B(p)\,\tilde{F}(p_1)\,\tilde{F}(p_2)\cdot 2$	$\tilde{G}'(p)\,\tilde{F}(p_1)\,\tilde{F}(p_2)\cdot 2$
$h^u_2(p\,p_1\,p_2\,p_3)$	$B(p)\,\tilde{F}(p_1)\,\tilde{F}(p_2)\,\tilde{F}(p_3)\cdot 6$	$\tilde{G}'(p)\,\tilde{F}(p_1)\,\tilde{F}(p_2)\,F(p_3)\cdot 6$.

(9.4)

$B(p)$ and $\tilde{G}'(p)$ are given by (7.21). One easily recognizes that the Eq. (7.20) can be obtained from (9.3) by summation over λ_1 i.e. the analytical structure has remained the same.

As in nonlinear spinor theory Eq. (9.3) is always used because of simplicity, we shall investigate for comparison the lowest approximations in the Dyson representation.

10. $N = 3$ approximation

The truncated equation system (9.3) for the $\tilde{\varphi}_1$ and $\tilde{\varphi}_3$-functions reads ($\varphi_3^1 \equiv \bar{\varphi}_3$!)

$$\tilde{\varphi}_1(p) = -\tilde{G}(p)\int \bar{\varphi}_3(p-\xi\,|\,\xi)\frac{\mathrm{d}\xi}{2\pi}, \tag{10.1}$$

$$\tilde{\varphi}_3(p_1 p_2 p_3) = 3\sum_{j\neq i,k} K^{\mathrm{u}}(p_i p_k)\,\bar{\varphi}_3(p_i+p_k\,|\,p_j) + \sum_{j\neq i,k} h^{\mathrm{u}}(p_i)\,2\pi\delta(p_i+p_k)\,\tilde{\varphi}_1(p_j) \tag{10.2}$$

$$+ 3h_{\mathrm{u}}^1(p_1 p_2 p_3)\,\tilde{\varphi}_1(p_1+p_2+p_3) \qquad (i=1,2,3).$$

Like in the preceding chapters we introduce again the linear combined function $\tilde{\chi}(p_1; p_2)$ given by (5.17) and get from Eq. (10.2)

$$\tilde{\varphi}_3(p_1 p_2 p_3) = \sum_{j\neq i,k}\{3K^{\mathrm{u}}(p_i p_k)\,\tilde{\chi}(p_i+p_k; p_j) + h^{\mathrm{u}}(p_i)\,2\pi\delta(p_i+p_k)\,\tilde{\varphi}_1(p_j)\} \qquad (i=1,2,3). \tag{10.3}$$

The application of the contraction operation (4.8) on the system (10.2) yields two different integral equations for the function $\tilde{\chi}(p_1; p_2)$

$$\tilde{\chi}(p_1; p_2)\,[1 - 3\bar{K}^{\mathrm{u}}(p_1)] \tag{10.4}$$

$$= 3\int K^{\mathrm{u}}(\xi - p_2, p_2)\,\tilde{\chi}(\xi; p_1+p_2-\xi)\frac{\mathrm{d}\xi}{2\pi} + \left[\frac{1}{2}A\,2\pi\delta(p_1) + \frac{1}{i}\tilde{G}(p_2)\right]\tilde{\varphi}_1(p_1+p_2)$$

for $i = 1, 2$,

$$\tilde{\chi}(p_1; p_2) = 6\int K^{\mathrm{u}}(p_2; \xi - p_2)\,\tilde{\chi}(\xi; p_1+p_2-\xi)\frac{\mathrm{d}\xi}{2\pi} - [\tilde{F}(p_2) + 2i\tilde{G}(p_2)]\,\tilde{\varphi}_1(p_1+p_2) \tag{10.5}$$

for $i = 3$.

By adding Eq. (10.4) once and Eq. (10.3) twice which has been multiplied by $\frac{1}{3}$ we obtain the symmetrized equation of Sect. 5. Defining

$$\gamma_1(p) =: [1 - 3\bar{K}^{\mathrm{u}}(p)]^{-1} \tag{10.6}$$

and substituting $p_1+p_2=:q$; $p_1=\eta$; $p_2=q-\eta$ the integral Eqs. (10.3) and (10.4) become — the structure is already wellknown from (5.9) and (8.8) —

$$\tilde{\chi}(\eta; q-\eta) = \gamma_1(\eta)\left[W_1(\eta; q-\eta) + 3\int K^{\mathrm{u}}(\xi - (q-\eta), q-\eta)\,\tilde{\chi}(\xi; q-\xi)\frac{\mathrm{d}\xi}{2\pi}\right] \tag{10.7}$$

$$\tilde{\chi}(\eta; q-\eta) = W_2(q-\eta) + 6\int K^{\mathrm{u}}(q-\eta, \xi-(q-\eta))\,\tilde{\chi}(\xi; q-\xi)\frac{\mathrm{d}\xi}{2\pi} \tag{10.8}$$

with the inhomogeneous parts

$$W_1(\eta; q-\eta) =: [\tfrac{1}{2}A\,2\pi\delta(\eta) + 1/i\,\tilde{G}(q-\eta)]\,\tilde{\varphi}_1(q), \tag{10.9}$$

$$W_2(q-\eta) =: -[\tilde{F}(q-\eta) + 2i\tilde{G}(q-\eta)]\,\tilde{\varphi}_1(q). \tag{10.10}$$

A simultaneous solution of both Eqs. (10.7) and (10.8) can hardly be found. Therefore we may suggest, to treat both equations separately, as has always been done in nonlinear spinor theory[3]. Then we have two different solutions $\tilde{\chi}_1$, $\tilde{\chi}_2$ and consequently two different eigenvalue equations where we can compare both possibilities by the resulting eigenvalues. The formal solutions of (10.7) and (10.8) are

$$\tilde{\chi}_1(\eta; q-\eta) = \gamma_1(\eta)\,[W_1(\eta; q-\eta) + \textstyle\int R_1(q-\eta; \xi)\,W_1(\xi; q-\xi)\,\mathrm{d}\xi/2\pi] \tag{10.11}$$

$$\tilde{\chi}_2(\eta; q-\eta) = W_2(q-\eta) + \textstyle\int R_2(q-\eta; \xi)\,W_2(q-\xi)\,\mathrm{d}\xi/2\pi \tag{10.12}$$

with the resolvents $R_1(q-\eta; \xi)$ and $R_2(q-\eta; \xi)$ belonging to the integral equation kernels

$$M_1(q-\eta; \xi) =: 3\,\gamma_1(\xi)\,K^{\mathrm{u}}(\xi - (q-\eta), q-\eta), \tag{10.13}$$

$$M_2(q-\eta; \xi) =: 6\,K^{\mathrm{u}}(q-\eta; \xi - (q-\eta)). \tag{10.14}$$

We use again the iterated form of Eq. (10.1) given as well by (5.23) and get the final eigenvalue equations

$$\frac{1}{3}[A + 3F(0) - \omega^2] + \frac{1}{2} A R_1^1(\omega;0) + \frac{1}{2\pi i}\int \tilde{G}(\omega-\xi) R_1^1(\omega;\xi)\,\mathrm{d}\xi = 0\,, \tag{10.15}$$

$$\frac{1}{3}[A + 3F(0) - \omega^2] - \frac{1}{2\pi}\int [\tilde{F}(\omega-\xi) + 2i\tilde{G}(\omega-\xi)] R_2^1(\omega;\xi)\,\mathrm{d}\xi = 0 \tag{10.16}$$

with the definitions

$$R_1^1(\omega;\xi) =: \bar{K}(\xi)\gamma_1(\xi) + \int \bar{K}(\eta)\gamma_1(\eta) R_1(\omega-\eta;\xi)\,\mathrm{d}\eta/2\pi\,, \tag{10.17}$$

$$R_2^1(\omega;\xi) =: \bar{K}(\xi) + \int \bar{K}(\eta) R_2(\omega-\eta;\xi)\,\mathrm{d}\eta/2\pi\,. \tag{10.18}$$

a) The approximate solutions of (10.15) and (10.16)

With the approximated functions $\tilde{F}$ app (p) and $\tilde{G}$ app (p) we have because of (9.4), (4.10) and (6.4)

$$\bar{K}^{\mathrm{u}}(p) = \frac{1}{2}\bar{K}(p) = \frac{(a+\omega_1)}{2a\omega_1}[p^2 - (a+\omega_1)^2 + i\varepsilon]^{-1} \tag{10.19}$$

and (10.6) becomes

$$\gamma_1(p) = \frac{p^2 - (a+\omega_1)^2}{p^2 - \gamma_1^2 + i\varepsilon} \quad \text{with} \quad \gamma_1^2 = (a+\omega_1)^2 + \frac{3}{2}\frac{(a+\omega_1)}{a\cdot\omega_1}\,. \tag{10.20}$$

We approximize again the resolvents R_1 and R_2 by its kernels (10.13) and (10.14) and thus get the final equations

$$\frac{1}{3}\frac{a\cdot\omega_1}{(a+\omega_1)}(A + a^2 - \omega^2) - \frac{1}{2}\frac{1}{\gamma_1^2}A + \frac{(a+\gamma_1)}{2a\gamma_1}[\omega^2 - (a+\gamma_1)^2]^{-1} + \frac{(-)}{2\pi i}\int g(\omega-\xi) N_1^{(1)}(\omega;\xi)\,\mathrm{d}\xi \equiv W_1(\omega;\omega_1) = 0\,, \tag{10.21}$$

$$\frac{1}{3}\frac{a\cdot\omega_1}{(a+\omega_1)}(A + a^2 - \omega^2) + \frac{(\omega_1+2a)}{a(\omega_1+a)}[\omega^2 - (\omega_1+2a)^2]^{-1} - \frac{(a+2\omega_1)}{2\omega_1(\omega_1+a)}[\omega^2 - (a+2\omega_1)^2]^{-1} + \frac{(-)}{2\pi i}\int [2g(\omega-\xi) - f(\omega-\xi)] N_2^{(1)}(\omega;\xi)\,\mathrm{d}\xi \equiv W_2(\omega;\omega_1) = 0 \tag{10.22}$$

with the definitions

$$N_1^{(1)}(\omega;\xi) =: 3\gamma_1(\xi)\frac{(-)}{2\pi i}\int \frac{g(\eta-\xi)f(\eta)}{[(\eta-\omega)^2 - \gamma_1^2 + i\varepsilon]}\,\mathrm{d}\eta\,, \tag{10.23}$$

$$N_2^{(1)}(\omega;\xi) =: 6\frac{(-)}{2\pi i}\int \frac{g(\eta)f(\eta-\xi)}{[(\eta-\omega)^2 - (a+\omega_1)^2 + i\varepsilon]}\,\mathrm{d}\eta\,. \tag{10.24}$$

The evaluation of these formulae and of the eigenvalue equations themselves are given in detail in App. I.

b) Special linear combination

For the calculation of the first eigenvalue ω_{10} a special linear combination of the equation system (10.2) is of special interest. Therefore we form the combinations $1+2-3$, and the cyclical permutations of them, of the corresponding Eqs. (10.2) with index $i = 1, 2, 3$ and get

$$\tilde{\varphi}_3(p_1 p_2 p_3) = 3K(p_i p_k)\tilde{\chi}(p_i + p_k; p_j) + 3\sum_{i\neq j} K_a(p_i p_j)\tilde{\chi}(p_i + p_j; p_k) + h(p_i)2\pi\delta(p_i + p_k)\varphi_1(p_j) \quad (j = 1,2,3) \tag{10.25}$$

with the definitions

$$K_a(p_1 p_2) = K^{\mathrm{u}}(p_1 p_2) - K^{\mathrm{u}}(p_2 p_1)\,, \qquad K(p_1 p_2) = K^{\mathrm{u}}(p_1 p_2) + K^{\mathrm{u}}(p_2 p_1)\,. \tag{10.26}$$

Application of the contraction operation then yields the integral equations (10.27)

$$\tilde{\chi}(p_1;p_2)[1 - 3\bar{K}(p_1)] = 6\int K_a(\xi - p_2; p_2)\tilde{\chi}(\xi; p_1 + p_2 - \xi)\,\mathrm{d}\xi/2\pi + [A\cdot 2\pi\delta(p_1) + \tilde{F}(p_2)]\tilde{\varphi}_1(p_1+p_2)$$

for $j = 3$,

$$\tilde{\chi}(p_1;p_2) = 6\int K^{\mathrm{u}}(p_2;\xi - p_2)\tilde{\chi}(\xi; p_1 + p_2 - \xi)\,\mathrm{d}\xi/2\pi - [\tilde{F}(p_2) + 2i\tilde{G}(p_2)]\tilde{\varphi}_1(p_1+p_2) \tag{10.28}$$

for $j = 1, 2$.

Equation (10.28) is identical with integral Eq. (10.4) obtained from the original Eq. (10.2), while (10.27) differs mainly from (10.3) in the kernel of the integral equation. Defining

$$\gamma_2(p) =: [1 - 3\bar{K}(p)]^{-1} \tag{10.29}$$

Eq. (10.27) becomes

$$\tilde{\chi}(p_1; p_2) = \gamma_2(p_1)\{6 \int K_a(\xi - p_2; p_2)\, \tilde{\chi}(\xi; p_1 + p_2 - \xi)\, \mathrm{d}\xi/2\pi + [A\, 2\pi\, \delta(p_1) + \tilde{F}(p_2)]\, \tilde{\varphi}_1(p_1 + p_2)\}\,. \tag{10.30}$$

Now $K_a(p_1\, p_2)$ vanishes for $f(p) \equiv g(p)$ identically. This means that in the lowest approximation (6.11) we can neglect the kernel K_a and get

$$\tilde{\chi}(p_1; p_2) = \gamma_2(p_1)\,[A\, 2\pi\, \delta(p_1) + \tilde{F}(p_2)]\, \tilde{\varphi}_1(p_1 + p_2)\,. \tag{10.31}$$

Using the approximated functions $f(p)$ and $g(p)$ Eq. (10.29) becomes with (6.4)

$$\gamma_2(p) = \frac{p^2 - (a + \omega_1)^2}{p^2 - \gamma_2^2 + i\varepsilon} \tag{10.32}$$

with $$\gamma_2^2 =: (a + \omega_1)^2 + 3\,\frac{(a + \omega_1)}{a \cdot \omega_1}$$

and (10.31) substituted in the iterated Eq. (5.23) leads to

$$W_3(\omega; \omega_1) \equiv \frac{1}{3}\,\frac{a \cdot \omega_1}{(a + \omega_1)}\,[A + a^2 - \omega^2] - \frac{A}{\gamma_2^2} + \frac{(\gamma_2 + \omega_1)}{2\gamma_2 \cdot \omega_1}\,[\omega^2 - (\gamma_2 + \omega_1)^2]^{-1} = 0\,. \tag{10.33}$$

V. Calculations and Results

a) Zero points

The approximated eigenvalue equations of the preceding chapters have been investigated numerically on the CDC 3200 computer of the University of Tübingen. In detail the following equations have been treated:

1. Dyson representation

$W(\omega; \omega_1) = 0$ according to (6.7) symmetrized, general equation

$W_0(\omega; \omega_1) = 0$ according to (6.15) symmetrized, lowest approximation

$W_1(\omega; \omega_1) = 0$ according to (10.21) not symmetrized, twofold weight

$W_2(\omega; \omega_1) = 0$ according to (10.22) not symmetrized, single weight

$W_3(\omega; \omega_1) = 0$ according to (10.33) not symmetrized, special linear combination, lowest approximation

2. Hermite representation

$W_H(\omega; \omega_1) = 0$ according to (8.21) symmetrized, general equation

$W_H^0(\omega; \omega_1) = 0$ according to (8.28) symmetrized, lowest approximation.

In structure, all these equations are linear combinations of meromorphic functions of the variable ω, the poles and coefficients of which are still depending on the parameter ω_1. This feature mainly characterizes the fact that the obtained integral equations have been solved by the iterative solution of a Neumann series, as discussed in app. II.

Equation	ω_1	z_1	z_2	z_3
Dyson	1.0000	0.9491		3.2886
W_0	1.0871	0.9790		3.4011
	1.1447	0.9962		3.4801
W_1	0.9086	1.1240	3.6470	4.0909
	1.0000	1.1010	3.7298	4.0537
	1.0871	1.0840	3.8060	4.0485
	1.1447	1.0748	3.8472	4.0667
W_2	0.9086	1.1511		3.7602
	1.0000	1.1451		3.7204
	1.0871	1.1377		3.6891
	1.1447	1.1325	3.1944	3.6620
W_3	0.9086	1.1192		4.1700
	1.0000	1.0947		4.2500
	1.0871	1.0767		4.3381
	1.1447	1.0672		4.4020
W	0.9086	1.1280	3.7793	4.0026
	1.0000	1.1077	3.7490	3.9617
	1.0871	1.0924	3.7589	3.9411
	1.1447	1.0840	3.6521	3.9369
Hermite	0.9086	0.8799		2.7351
W_H^0	1.0000	0.9243		2.9419
	1.0871	0.9672		3.1462
	1.1447	0.9962		3.2845
W_H	0.9086	0.8903	2.8279	2.9196
	1.0000	0.9280	2.9676	3.0248
	1.0871	0.9674	3.1112	3.1637
	1.1447	0.9951	3.2118	3.2592

Table 1. Zero points z_i of the equations $W_0(\omega; \omega_1) = 0, \ldots, W_H(\omega; \omega_1) = 0$ for the parameter values ω_1.

The zero points $z_i(\omega_1)$ of the various eigenvalue equations have been determined in the ω-interval [0.5, 6.0] with an accuracy of 10^{-7} as function of the parameter ω_1. The value of the parameter ω_1 itself runs trough the interval [0.7, 1.3] with interval steps of 0.02. As discussed in Chapter 6.A the following fixed ω_1-values are of special interest

$$\omega_1 = \begin{array}{l} 0.9085 \text{ Hermite } N=1 \text{ approx. (8.25)} \\ 1.0000 \text{ harmonic oscillator} \\ 1.0871 \text{ exact value with Schrödinger theory} \\ 1.1447 \text{ Dyson } N=1 \text{ approx. (6.11).} \end{array}$$

For them we get zero points arranged in Table 1.

b) Identification of zero points with physical eigenvalues

We expect only to get the physical eigenvalues ω_{10} and ω_{30}; and it is the question which zero point has to be identified with them, because with each higher iterative approximation of the corresponding integral equations we get more physically irrelevant zero points as the degree of the resulting algebraic equations increases. We use therefore some principles which should be satisfied.

1. Identification of the lowest positiv zero point z_1 with the lowest physical eigenvalue ω_{10} as long as this does not contradict conditions 2 and 3.

2. Maximal consistence against variation of the parameter ω_1 i.e. such zero points are of no importance which vary too much as function of ω_1 or which even disappear or appear. This condition is a consequence of the invariance of the physical eigenvalues against the choice of different base functionals.

3. Consistence against the various equations i.e. only such solutions of the individual equations are acceptable which satisfy approximatively the original equation system as discussed in Chapter IV. This means that only such zero points are of interest which show nearly the same value in all equations and in the neighbourhood of which corresponding equations show a similar functional behaviour as functions of the variable ω in fixed parameter ω_1.

With the aid of these criteria we can identify the zero points of Table 1 in the following way: $z_1 \equiv \omega_{10}$; $z_3 \equiv \omega_{30}$. The quantity z_2 has to be rejected namely in detail because of

equation	W_1	W_2	W	W_H
rejection condition	2	2	3	3

c) The selfconsistent eigenvalue Ω_{10}

This value is determined by the condition

$$z_1(\omega_1) \equiv \omega_1$$

and it is thus obtained in full analogy to calculations in nonlinear spinor theory as we have there no information about the parameter ω_1 (exept values obtained in lower approximations).

d) Results

With this the calculated eigenvalues $\omega_{10}(\omega_1)$, $\omega_{30}(\omega_1)$ and the selfconsistent values Ω_{10} can finally be given in Table 2 and 3. In addition also those values are given, which we can get by arithmetic averaging of the values obtained from Eqs. W_1 and W_2. They agree quite well to those obtained from Eq. W which has already been averaged before. Furthermore the relative deviations Δ_{10} and Δ_{30} from the corresponding exact value are stated at the selfconsistent eigenvalue Ω_{10}. From the follow-

ω_1	0.9086	1.0000	1.0871	1.1447	Ω_{10}	Δ_{10} [%]
Dyson						
W_0		0.9491	0.9790	0.9962	0.9128	+16.1
W_1	1.1240	1.1010	1.0840	1.0748	1.0844	− 0.3
W_2	1.1511	1.1451	1.1377	1.1325	1.1335	+ 4.2
W_3	1.1192	1.0947	1.0767	1.0672	1.0783	− 0.8
W	1.1280	1.1077	1.0924	1.0840	1.0917	+ 0.4
$\frac{1}{3}(2W_1+W_2)$	1.1330	1.1157	1.1016	1.9040	1.1008	+ 1.3
Hermite						
W_H^0	0.8799	0.9243	0.9672	0.9962	0.8527	−21.6
W_H	0.8903	0.9280	0.9674	0.9951	0.8791	−19.2

Table 2. Eigenvalue ω_{10}. Exact value $E_{10} = 1.0871$; $\Delta_{10} = : (\Omega_{10} - E_{10})/E_{10}$.

ω_1	0.9086	1.0000	1.0871	1.1447	Ω_{10}	Δ_{30} [%]
Dyson						
W_0		3.2886	3.4011	3.4801	3.1861	−24.1
W_1	4.0909	4.0537	4.0485	4.0667	4.0482	− 3.6
W_2	3.7602	3.7204	3.6891	3.6620	3.6687	−12.6
W_3	4.1700	4.2500	4.3381	4.4020	4.3287	+ 3.1
W	4.0026	3.9617	3.9411	3.9369	3.9405	− 6.2
$\frac{1}{3}(2W_1+W_2)$	3.9817	3.9759	3.9287	3.9318	3.9217	− 6.7
Hermite						
W_H^0	2.7351	2.9419	3.1462	3.2845	2.6136	−38
W_H	2.9196	3.0248	3.1637	3.2592	2.8830	−32

Table 3. Eigenvalue ω_{30}. Exact value $E_{30} = 4.2002$; $\Delta_{30} = (\omega_{30}(\Omega_{10}) - E_{30})/E_{30}$.

ing Fig. 1 and 2 the course of the eigenvalues $\omega_{10}(\omega_1)$ and $\omega_{30}(\omega_1)$ as functions of the parameter ω_1 can be seen for the various equations.

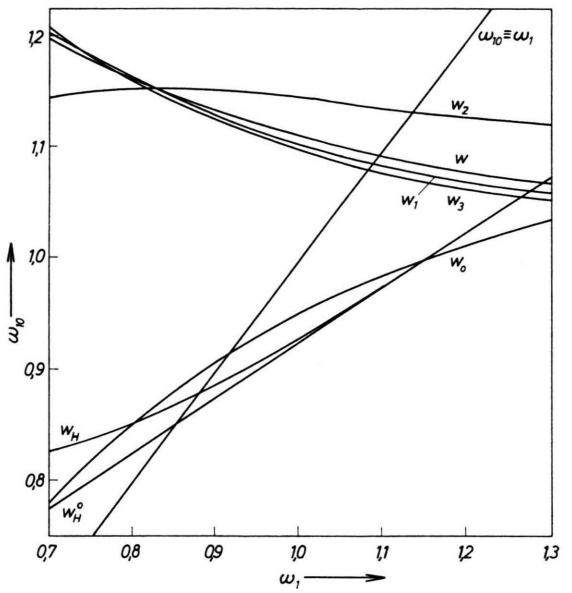

Fig. 1. Eigenvalue $\omega_{10}(\omega_1)$ as function of the parameter ω_1 for the various equations $W_0 \ldots W_H$.

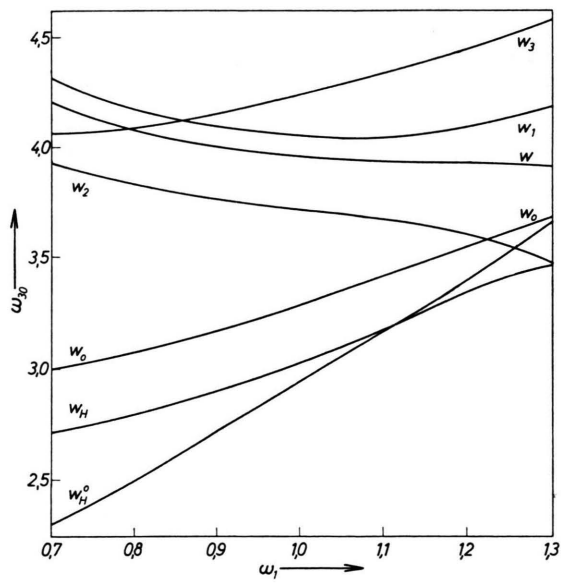

Fig. 2. Eigenvalue $\omega_{30}(\omega_1)$ as function of the parameter ω_1 for the various equations $W_0 \ldots W_H$.

Summarizing the results of our calculations we state:

1. In the investigated q-case the Hermite representation leads to completely unacceptable values especially as far as the higher eigenvalues are concerned. This is not surprising as the smeared out functional Eq. (2.6) is not Hermitean (in the sense of functional analysis) in contrast to the $p-q$ case[14]. But only for a Hermitean equation we can hope to improve the eigenvalues, as then this quality is preserved in the symmetric Hermite representation but not in the Dyson one.

2. The Dyson representation, however, results in very good eigenvalues. Here the unsymmetrized Eq. W_1 and the symmetrized Eq. W yield nearly the same good values, whereas the other unsymmetrized Eq. W_2 results in somewhat worse ones. (In nonlinear spinor theory it is supposed, that this type of equation cannot be used in the $N=3$ approximation!) It turns out that the appropriately chosen linear combination W_3 of the unsymmetrized equations leads to excellent results in simplest approximation without much calculational effort.

3. We have to apply various equations instead of only one in order to be able to identify the obtained zero points with physical eigenvalues as long as we are only considering low N.T.D. approximations. Additional group theoretic arguments can not be used. On the other hand it is practically impossible to calculate a set of eigenvalues ω_N from higher N.T.D. approximations in order to reject the accidental unphysical zero points.

Acknowledgements

The author is very indebted to Prof. Dr. H. STUMPF for his kind and critical reading of the manuscript. He also wishes to thank very much Prof. Dr. W. HEISENBERG and Dr. H. P. DÜRR of the Max-Planck-Institut für Physik and Astrophysik, Munich, for their interest in this subject. He gratefully acknowledges the financial support of the Deutsche Forschungsgemeinschaft.

Appendix I

Here we shall take a glance at the used approximated *two point function* $f(p)$ and *Green's function* $g(p)$ as given by (6.1) and (6.2). They are both functions of the kind

$$f(p;c_k) =: [p^2 - c_k^2 + i\varepsilon]^{-1} \quad c_k > 0\,, \quad \text{real}$$
$$\equiv [p - c_k + i\varepsilon']^{-1}[p + c_k - i\varepsilon]^{-1} \quad \text{(I.1)}$$

just as the exact functions $\tilde{F}(p)$ and $\tilde{G}(p)$. These functions can only be integrated in Feynman's sense i.e. the integration is carried out at first with finite ε (or ε') and only after calculation the limes $\varepsilon \to 0$

is made. But this means that we are dealing with distributions. In the language of well known distributions (I.1) can be written as

$$f(p; c_k) = \frac{\pi}{i c_k}[\delta_+(p - c_k) + \delta_-(p + c_k)] \quad \text{(I.2)}$$

where δ_+ and δ_- are the positive and negative frequency part of Dirac's $\delta(p)$ function[23]. But Feynman's prescription makes also sense for products of functions of type (I.1) if handled with caution and thus defines a product theory of distributions of type (I.2). For example the convolution of two functions $f(p, c_1)$ and $f(p, c_2)$ does not lead out of the space $\mathscr{D}$ of functions of type (I.1). We have

$$\frac{(-)}{2\pi i}\int f(p-\xi; c_1) f(\xi; c_2)\,\mathrm{d}\xi = \frac{(c_1 + c_2)}{2 c_1 c_2} f(p; c_1 + c_2) \quad \text{(I.3)}$$

and especially

$$\frac{(-)}{2\pi i}\int g(p-\xi) f(\xi)\,\mathrm{d}\xi = \frac{(a+\omega_1)}{2 a \omega_1}[p^2 - (a+\omega_1)^2 + i\varepsilon]^{-1}. \quad \text{(I.4)}$$

Now we shall specify the *eigenvalue equations* in detail. We begin with eq. $W(\omega; \omega_1) = 0$ according (6.7). Then the evaluation of formula (6.9) for

---

$N^{(1)}(\omega; \eta)$ and partial decomposition leads to

$$N^{(1)}(\omega; \eta) = \frac{1}{2}\gamma(\eta)\left[\frac{\alpha_1}{\eta - \omega - c_1} + \frac{\bar\alpha_1}{\eta - \omega - \bar c_1} - \frac{\alpha_2}{\eta - \omega + c_1} - \frac{\bar\alpha_2}{\eta - \omega + \bar c_1} + \frac{\alpha_3}{\eta - c} - \frac{\alpha_4}{\eta + c}\right] \quad \text{(I.5)}$$

with the definitions (by a bar the permutation $\omega_1 \to a$ is indicated)

$$\begin{aligned} &c_1 =: \gamma + a - i\varepsilon, \qquad \bar c_1 =: \gamma + \omega_1 - i\varepsilon, \qquad c =: \omega_1 + a - i\varepsilon, \\ &\alpha_1 =: \frac{1}{\gamma \cdot a} f(\gamma + \omega), \quad \bar\alpha_1 =: \frac{1}{\gamma \cdot \omega_1} g(\gamma + \omega), \quad \alpha_3 =: \frac{1}{\omega_1 a}[\hat\gamma(\omega - \omega_1) + \hat\gamma(\omega - a)], \\ &\alpha_2 =: \frac{1}{\gamma \cdot a} f(\gamma - \omega), \quad \bar\alpha_2 =: \frac{1}{\gamma \cdot \omega_1} g(\gamma - \omega), \quad \alpha_4 =: \frac{1}{\omega_1 a}[\hat\gamma(\omega + \omega_1) + \hat\gamma(\omega + a)], \\ &\hat\gamma(p) =: [p^2 - \gamma^2 + i\varepsilon]^{-1}. \end{aligned} \quad \text{(I.6)}$$

Then we get after some algebra

$$\begin{aligned} \underline{W(\omega; \omega_1)} \equiv\ & \frac{a\omega_1}{(a+\omega_1)}(a^2 - \omega^2) + \frac{\omega_1^2 - a^2}{\gamma^2} + \frac{2(a+\gamma)}{a\cdot\gamma}\gamma(0)[\omega^2 - (a+\gamma^2]^{-1} \\ &+ \frac{(\omega_1+\gamma)}{2\omega_1\gamma}\left[\frac{4(a+\omega_1)}{\omega_1 \cdot a\gamma^2} - 1\right][\omega^2 - (\omega_1 + \gamma)^2]^{-1} \\ &+ \frac{1}{a}\hat\gamma(\omega + a)\left[(\omega_1 - \omega)(\omega + \omega_1 + 2a)\left[\frac{\alpha_1}{\gamma} + \frac{\bar\alpha_1}{\gamma + \omega_1 - a} + \frac{\alpha_2}{\gamma + 2a} + \frac{\bar\alpha_2}{\gamma + a + \omega_1}\right] + (\omega + \omega_1 + 2a)\alpha_3\right. \\ &\left. + (\omega_1 - \omega)\alpha_4\right] - \frac{1}{4\omega_1}\hat\gamma(\omega + \omega_1)\left[(a - \omega)(\omega + a + 2\omega_1)\left[\frac{\bar\alpha_1}{\gamma} + \frac{\alpha_1}{\gamma + a - \omega_1} + \frac{\bar\alpha_2}{\gamma + 2\omega_1} + \frac{\alpha_2}{\gamma + a + \omega_1}\right]\right. \\ &\left. + (\omega + a + 2\omega_1)\alpha_3 + (a - \omega)\alpha_4\right] + \frac{1}{2\gamma}\left[2g(\omega - \gamma) - \frac{1}{2}f(\omega - \gamma)\right]\left[-\frac{(a+\omega_1)}{a\cdot\omega_1}\left[\frac{\alpha_1}{\omega + a} + \frac{\bar\alpha_1}{\omega + \omega_1}\right.\right. \\ &\left.\left. + \frac{\alpha_2}{2\gamma + a - \omega} + \frac{\bar\alpha_2}{2\gamma + \omega_1 - \omega}\right] + \alpha_3(c + \gamma) + \alpha_4(c - \gamma)\right] + \alpha_1\gamma(\omega + \gamma + a)\left[2g(\gamma + a) - \frac{1}{2}f(\gamma + a)\right] \\ &+ \bar\alpha_1\gamma(\omega + \gamma + \omega_1)\left[2g(\gamma + \omega_1) - \frac{1}{2}f(\gamma + \omega_1)\right] = 0. \end{aligned} \quad \text{(I.7)}$$

Analogously we proceed with the equations

$$W_1(\omega; \omega_1) = 0, \quad W_2(\omega; \omega_1) = 0$$

and $W_H(\omega; \omega_1) = 0$ given by (10.21), (10.22) and (8.21). We then get equations of the same structure.

[23] W. GÜTTINGER, Fortschr. Physik **14**, 483 [1966].

Appendix II

In this appendix the *singular integral equations* shall be discussed. With the definition

$$\Psi_q(\eta) =: \tilde\chi(\eta; q - \eta) \quad \text{(II.1)}$$

they have the structure

$$\Psi_q(\eta) = \int M(q - \eta; \xi)\,\Psi_q(\xi)\,\frac{\mathrm{d}\xi}{2\pi} + S_q(\eta) \quad \text{(II.2)}$$

q being an implicit parameter.

The special kernels are given by

$$M(q-\eta;\xi)=\begin{cases}2\gamma(\xi)\,K(\xi-(q-\eta),q-\eta) & \text{(II.3a)}\\ 2\gamma_{\mathrm{H}}(\xi)\,K_{\mathrm{H}}(\xi-(q-\eta),q-\eta) & \text{(II.3b)}\\ 3\gamma_1(\xi)\,K^{\mathrm{u}}(\xi-(q-\eta),q-\eta) & \text{(II.3c)}\\ 6\,K^{\mathrm{u}}(q-\eta,\xi-(q-\eta)) & \text{(II.3d)}\end{cases}$$

and the inhomogenous parts by (5.19), (8.9), (10.9) and (10.10). Remembering the definition of the different functions $\gamma(p)$ and $K(p_1,p_2)$ we can see that the kernels (II.3) are mainly built up by $\tilde{F}(p)$ and $G(p)$ functions i.e. by distributions of the kind (I.1) and (1.2) respectively. This is a general feature of dynamical calculations in quantum field theory as all characteristic functions such as propagators, commutators, Green's functions etc. have this structure[24]. Thus we are consequently led to integral equations with Feynman kernels. Using the approximation functions $f(p)$ and $g(p)$ for $\tilde{F}$ and $\tilde{G}$ the kernels (II.3) become in the notation of (I.1)

$$M(q-\eta;\xi) \qquad \text{(II.4a)}$$
$$=2f^{-1}(\xi;a+\omega_1)f(\xi;\gamma)\,[f(q-\eta-\xi;\omega_1)f(q-\eta;a)+f(q-\eta-\xi;a)f(q-\eta;\omega_1)]$$

$$\tfrac{1}{2}f^{-1}(\xi;b+\omega_1)f(\xi;\gamma_{\mathrm{H}})\,[f(q-\eta-\xi;\omega_1)f(q-\eta;b)+f(q-\eta-\xi;b)f(q-\eta;\omega_1)], \qquad \text{(II.4b)}$$

$$3f^{-1}(\xi;a+\omega_1)f(\xi;\gamma_1)f(q-\eta-\xi;a)f(q-\eta;\omega_1), \qquad \text{(II.4c)}$$

$$6f(q-\eta-\xi;\omega_1)f(q-\eta;a) \qquad \text{(II.4d)}$$

with the constants γ, γ_{H}, γ_1 given by (6.5), (8.17) and (10.20). Because of this the integral Eqs. (II.1) have only to be considered in the distribution sense; all terms of an iterative series are declared, as the convolution of the functions $f(p;c_k)$ exists according to (I.3) and gives functions of the same structure. Thus any iterative solution does not lead out of the linear space $\mathscr{D}$ of these distributions of kind (I.1). This means, that the Neumann series

$$R(q-\eta;\xi)=\sum_{n=1}^{\infty}M^n(q-\eta;\xi) \qquad \text{(II.5)}$$

with

$$M^n(q-\eta;\xi)=\int M^{n-1}(q-\eta;x)\,M(q-x;\xi)\,\mathrm{d}x$$
$$M^1(q-\eta;\xi)=:M(q-\eta;\xi)$$

lies also in $\mathscr{D}$. However, the main problem is to show the convergence in distribution sense and to fix the q spectrum, for a suitably chosen test function space.

Unfortunately, no systematic treatment of integral equations of this kind has been given although they are already known in quantum field theory for a long time. Even in the simplest case of a relativistic two particle equation like the Bethe-Salpether equation[25a] only tricks have been applied to handle the problem; and these tricks contain dangerous assumptions not really justified. Even the well known Wick rotation[25a] has not been proved rigorously.

The reason for this lack of rigour lies in the fact, that the Feynman kernels are distributions of kind (I.1) and do not belong to a class of L^2 functions i.e. the well known Hilbert space methods are not applicable[25c]. The integral equations defined with them may rather be called "super-Cauchy" equations due to the character of their singularities. And although integral equations of the simpler Cauchy type have been investigated intensively[25b,c] nothing is known about these "super-Cauchy" equations. In a paper of SCHULER and STUMPF[25a] the simplest case of the integral equations (I.2) has been treated with the kernel (II.3d) for $q=0$, but a complete theory is still open. It may be hoped that these Feynman integral equations can be mastered with the method of distribution theory[23,24] and the concept of rigged Hilbert spaces[26].

[24] E. PFAFFELHUBER, Thesis, University of Munich 1967.

[25] a) W. SCHULER and H. STUMPF, preprint, University of Tübingen 1968. — b) N. I. MUSKELISHVILI, Singular Integral Equations, Noordhoff, Gronningen 1967. — c) W. POGORZELSKI, Integral Equations and their Applications, Pergamon Press, New York 1966.

[26] I. M. GELFAND and N. J. WILENKIN, Verallgemeinerte Funktionen IV, Deutscher Verlag der Wissenschaften, Berlin 1964.